SS 201A, SET 2
DUE THURSDAY, NOVEMBER 18TH

Collaboration on homework is encouraged, but individually written solutions are required. Please name all collaborators and sources of information on each assignment. Any such named source may be used.

(1) *Symmetric equilibria.* A normal form game with two players $G = (\{1,2\},(S_1,S_2),(u_1,u_2))$ is *symmetric* if $S_1 = S_2$ and for every s_1,s_2 it holds that $u_1(s_1,s_2) = u_2(s_2,s_1)$. A mixed strategy profile (σ_1,σ_2) is symmetric if $\sigma_1 = \sigma_2$. A correlated equilibrium μ is symmetric if $\mu(s_1,s_2) = \mu(s_2,s_1)$ for all s_1,s_2.

(a) Give an example of a symmetric G that has no symmetric pure equilibria.

(b) Prove (without using Brouwer, Nash etc) that every symmetric G has a symmetric mixed equilibrium. Assume $|S_1| = |S_2| = 2$.

(c) Show that in a symmetric game there is always a symmetric correlated equilibrium among the correlated equilibria that maximize the sum of expected utilities

(2) *The surprise quiz.* A teacher and a student play the following game. The teacher gives a surprise quiz on one of the five days of the work week. The student, who does not know the material, will fail if he does not review the material right before the quiz, but only has time to study on one day. Thus each player's set of the strategies is the set of five days of the work week. The student's utility is one if he and the teacher chose the same day, and zero otherwise. The teacher's utility is one minus the student's.

(a) Show that this game does not have a pure Nash equilibrium.

(b) Find a mixed equilibrium for this game.

(c) Show that if there are (countably) infinitely many days then there does not exist a mixed equilibrium.

(3) *Bonus question: Prisoner's Escape.* A prisoner escapes to the number line. He chooses some $n \in \mathbb{Z}$ to hide on the zeroth day. He also chooses some $k \in \mathbb{Z}$, and every day hides at a number that is k higher than in the previous day. Hence on day $t \in \{0,1,2,\ldots\}$ he hides at $n + k \cdot t$.

Every day the detective can check one number and see if the prisoner is there. If he is there, she wins. Otherwise she can check again the next day.

Formally, the game played between the prisoner and the detective is the following. The prisoner's strategy space is $\{(n,k) : n,k \in \mathbb{Z}\}$, and the detective's stragegy space is the set of sequences $(a_0,a_1,a_2,\ldots)$ in $\mathbb{Z}$. The detective wins if $a_t = n + k \cdot t$ for some t. The prisoner wins otherwise.

Prove that the detective has a winning strategy.

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(4) Let G be a finite extensive form game with perfect information in which each player plays at one history only. That is, for each i there is a single history $h \in H$ such that $P(h) = i$.

(a) Prove: every pure trembling hand perfect equilibrium of the strategic form of G is a subgame perfect equilibrium of G.

(b) Prove or disprove: every pure subgame perfect equilibrium of a finite extensive form game with perfect information is a trembling hand perfect equilibrium of the strategic form of the game.

(5) Construct an example of a knowledge space with two players, a finite set of states of the world, an event A and a state of the world ω such that $\omega \in K_1A$, $\omega \in K_2A$, $\omega \in K_1K_2A$, $\omega \in K_2K_1A$, but $\omega \notin K_1K_2K_1A$.

(6) For $\Omega = \{1,2\}$, construct an operator $L\colon 2^\Omega \to 2^\Omega$ that satisfies the first four Kripke axioms, but not the last. Propose a setting that you think is well modeled by L.

(7) For $\Omega = \mathbb{Z}$, construct an example of a knowledge space for two players such that each information set is of size two, and no event other than Ω is common knowledge at at $\omega \in \Omega$.

(8) *Bonus question*. A prisoner escapes to $\mathbb{Z}^2$ on Sunday. Every day he must move either one up (i.e., add $(0,1)$ to his location) or one to the right (add $(1,0)$), except on Saturdays, when he must rest. The detective can, once a day, check one element of $\mathbb{Z}^2$ and see if the prisoner is there. If she finds him then she wins. He wins if she never finds him.

Formally, the prisoner's strategy is an element

$$(z,f) \in \mathbb{Z}^2 \times \{(1,0),(0,1),(0,0)\}^{\mathbb{N}}$$

such that $f(n) = (0,0)$ whenever $n \equiv 0 \mod 7$, and $f(n) \in \{(1,0),(0,1)\}$ otherwise. The detective's strategy is a sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n \in \mathbb{Z}^2$.

The prisoner's current location when using strategy (z,f) is

$$\ell_n = z + \sum_{k=1}^{n-1} f(k).$$

The detective wins if $\ell_n = z_n$ for some n. The prisoner wins otherwise.

(a) Show that the detective has a winning strategy.

(b) Show that if we remove the requirement that the prisoner rests on Saturdays then the detective does not have a winning strategy.