Invariant random subgroups of semidirect products

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Abstract

We study invariant random subgroups (IRSs) of semidirect products $G = A \rtimes \Gamma$. In particular, we characterize all IRSs of parabolic subgroups of $\mathrm{SL}_d(\mathbb{R})$, and show that all ergodic IRSs of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$ are either of the form $\mathbb{R}^d \rtimes K$ for some IRS of $\mathrm{SL}_d(\mathbb{R})$, or are induced from IRSs of $\Lambda \rtimes \mathrm{SL}(\Lambda)$, where $\Lambda < \mathbb{R}^d$ is a lattice.

Contents

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1 Introduction

Let G be a locally compact, second countable group and let Sub_G be the space of closed subgroups of G, considered with the Chabauty topology [10].

Definition 1. An *invariant random subgroup* (IRS) of G is a random element of Sub_G whose law is a conjugation invariant Borel probability measure.

IRSs were introduced by Abért–Glasner–Virág in [2], and independently by Vershik [22] (under a different name) and the second author [7]. Examples of IRSs include normal subgroups, as well as random conjugates $g\Gamma g^{-1}$ of a lattice $\Gamma < G$, where the conjugate is chosen by selecting Γg randomly against the given finite measure on $\Gamma\backslash G$. More generally, any IRS of a lattice $\Lambda < G$ *induces* an IRS of G: if μ_Γ is the law of the original IRS and η is a G-invariant probability measure on $\Gamma\backslash G$, the new law μ_G is given by the integral

$$\mu_G = \int_{\Gamma g \in \Gamma\backslash G} g_* \mu_\Gamma \, d\eta,$$

where μ_Γ is regarded as a measure on $\mathrm{Sub}_\Gamma \subset \mathrm{Sub}_G$, and g acts on Sub_G by conjugation. Informally, we conjugate the IRS of Γ by an 'η-random' element of G. Since Sub_G is compact [5, Lemma E.1.1], the space of (conjugation invariant) Borel probability measures on Sub_G is weak* compact, by Riesz's representation theorem and Alaoglu's theorem. Hence, IRSs compactify the set of lattices in G. There is a growing literature on IRSs (see, e.g., [3, 6, 8, 9, 19]) and their applications, see especially [1, 7, 13, 21].

Our goal in this note is to develop an understanding of IRSs of semidirect products $G = A \rtimes \Gamma$. There are few general constructions of such IRSs: there is the trivial IRS $\{e\}$, and IRSs of the form $A \rtimes K$, where K is an IRS of Γ. When the kernel Γ_{triv} of the action $\Gamma \circlearrowright A$ is nontrivial, one can also construct IRSs of the form $H \rtimes K$, where H is an IRS of A and K is an IRS of Γ that lies in Γ_{triv}, but additional examples are hard to find.

The kernel of our work are Theorems 2.6 and 2.7, in which we study 'transverse' IRSs of $G = A \rtimes \Gamma$ when A is torsion-free abelian or simply connected nilpotent. Here, an IRS $H < G$ is *transverse* if $H \cap A = \{0\}$. This theorem has two parts: when A is torsion-free abelian, we prove that that the projection of H to Γ acts trivially on A almost surely, and if A is a simply connected nilpotent Lie group, we show that an (often large) subgroup of Γ acts precompactly on the Zariski closure of the set of all first coordinates of elements $(v, M) \in H$, as H ranges through the support of the IRS.

As applications of Theorems 2.6 and 2.7, we study IRSs of two familiar semidirect products: the *special affine groups* $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$ and the *parabolic subgroups* of $\mathrm{SL}_d(\mathbb{R})$.

1.1 IRSs of special affine groups

We are particularly interested in IRSs of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$. In addition to the examples $\{e\}$ and $\mathbb{R}^d \rtimes K$ mentioned above, one can construct an IRS from a lattice $\Lambda \subset \mathbb{R}^d$. Namely, the subgroup $\mathrm{SL}(\Lambda) < \mathrm{SL}_d(\mathbb{R})$ stabilizing Λ is also a lattice, see [17], so the semidirect product $\Lambda \rtimes \mathrm{SL}(\Lambda)$ is a lattice in $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$, and hence a random conjugate of it is an IRS.

Theorem 1.1. *Let H be a non-trivial ergodic IRS of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$. Then either*

1. *$H = \mathbb{R}^d \rtimes K$ for some IRS $K < \mathrm{SL}_d(\mathbb{R})$, or*
2. *H is induced from an IRS of $\Lambda \rtimes \mathrm{SL}(\Lambda)$, for some lattice $\Lambda < \mathbb{R}^d$.*

Here, an IRS is *ergodic* if its law is an ergodic measure for the conjugation action of G on Sub_G. By Choquet's theorem [16], every IRS can be written as an integral of ergodic IRSs. Note that by transitivity of the action of $\mathrm{SL}_d(\mathbb{R})$ on the space of lattices of a fixed covolume, we can actually choose Λ in 2. to be a scalar multiple of $\mathbb{Z}^d$.

As a corollary, any normal subgroup of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$ is of the form $\mathbb{R}^d \rtimes K$ where K is a normal subgroup of $\mathrm{SL}_d(\mathbb{R})$. (Here, $K = \{e\}$, $\mathrm{SL}_d(\mathbb{R})$ or $\{\pm I\}$, where the last option is only available when d is even.) Similarly, it follows that every lattice of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$ is a finite index subgroup of some $\Lambda \rtimes \mathrm{SL}(\Lambda)$. We expect that these results are not entirely surprising, although we note that Theorem 4.8 of [11] is that $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$ has no uniform lattices, which follows trivially from this classification.

Stuck–Zimmer [18] show that for $d > 2$, every ergodic IRS of $\mathrm{SL}_d(\mathbb{R})$ is either a lattice or a normal subgroup. This result, together with Theorem 1.1, implies that for $d > 2$ every ergodic IRS of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$ is likewise either a lattice or a normal subgroup.

In light of Theorem 1.1, to understand IRSs in special affine groups it suffices to study those of $G = \mathbb{Z}^d \rtimes \mathrm{SL}_d(\mathbb{Z})$. There are the usual examples $\{e\}$ and $\mathbb{Z}^d \rtimes K$, where K is an IRS of $\mathrm{SL}_d(\mathbb{Z})$, but in general, some subtle finite group theory appears. For instance, let

$$\pi_n : G \longrightarrow (\mathbb{Z}/n\mathbb{Z})^d \rtimes \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$$

be the reduction map and setting $d = 2$, consider the subgroup

$$H = \left\{ \left((t,0), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^t \right) \;\middle|\; t \in \mathbb{Z}/n\mathbb{Z} \right\} < (\mathbb{Z}/n\mathbb{Z})^d \rtimes \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z}).$$

The preimage $\pi_n^{-1}(H)$ is a finite index subgroup of G, and therefore can be considered as an IRS, but it does not have the form $\Lambda \rtimes K$ for any $\Lambda < \mathbb{Z}^d, K < \mathrm{SL}_d(\mathbb{Z})$. However, we will show that all IRSs of G are semidirect products up to some 'finite index noise'. Namely, let

$$G_n = Ker\, \pi_n = n\mathbb{Z}^d \rtimes \Gamma(n),$$

where $\Gamma(n)$ is the kernel of the reduction map $\mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$. We prove:

Theorem 1.2. *Let H be a non-trivial ergodic IRS of $\mathbb{Z}^d \rtimes \mathrm{SL}_d(\mathbb{Z})$. Then there is some $n \in \mathbb{N}$ such that $H_n = H \cap G_n$ is of the form $n\mathbb{Z}^d \rtimes K$, where K is an IRS of $\mathrm{SL}_d(\mathbb{Z})$.*

Remark 1. In the case of $G = \mathrm{SL}_n(\mathbb{Z})$, the Nevo–Stuck–Zimmer theorem says that any ergodic IRS of G is either finite index almost surely, or is central in G, see [18, 15]. Bekka [4] later generalized this to a rigidity statement about the *characters* of G. Here, an IRS with law μ gives the character $\phi : G \longrightarrow [0,1]$, where $\phi(g) = \mu(\{H \in \mathrm{Sub}_G \mid g \in H\})$.

Specializing Bekka's proof to the case of IRSs, Theorem 1.2 can be used in place of his Sections 4 and 5 (and a bit of 6) in a fairly elementary proof of the Nevo–Stuck–Zimmer theorem for $G = \mathrm{SL}_n(\mathbb{Z})$. Namely, suppose $H \le G$ is an ergodic infinite index IRS. Writing

$$\mathbb{Z}^n = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle,$$

we can let $P_i \cong \mathbb{Z}^{n-1} \rtimes \mathrm{SL}_{n-1}(\mathbb{Z})$ be the parabolic subgroup of G that is the stabilizer of x_i, and let $V_i \cong \mathbb{Z}^{n-1}$ be the corresponding unipotent subgroup of P_i. Theorem 1.2 says that for each i, either $H \cap P_i$ is almost surely trivial or H almost surely contains a lattice in V_i. If for every i, we have that $H \cap V_i$ is a lattice in V_i a.s., then there is some m such that a.s. H contains the m^{th} powers of all elementary matrices, which implies H is finite index, e.g. by Tits [20]. So, we can assume that for some i, $H \cap P_i$ is trivial. Similarly, we can assume that $H \cap P_j^t$ is trivial for some j, where P_j^t is the parabolic subgroup one gets by taking the transposes of all the matrices in P_j. Moreover, we can assume $i \neq j$, since if P_i and P_i^t were the only parabolics intersecting H trivially, one would still get all possible m^{th} powers of elementary matrices in H as above. Switching indices so that $(i, j) = (n, 1)$ puts us at the beginning of Bekka's Section 6—and in fact, we already know Lemma 15.

This gives a proof of the Nevo–Stuck–Zimmer theorem for $\mathrm{SL}_n(\mathbb{Z})$ in which the only ingredients are our Theorem 1.2 (which is actually easier to prove than much of the content of this paper), the fact that the m^{th} powers of all the elementary matrices generate a finite index subgroup of $\mathrm{SL}_n(\mathbb{Z})$, and the elementary arguments in [4, §6].

1.2 IRSs of parabolic subgroups of $\mathrm{SL}_d(\mathbb{R})$

Suppose that $W = \mathbb{R}^d$ is a finite dimensional real vector space, written as a direct sum

$$W = S_1 \oplus \cdots \oplus S_n$$

of subspaces, and that $\mathcal{F}$ is the associated flag

$$0 = W_0 < W_1 < \cdots < W_n = W, \quad W_k = \oplus_{i=1}^k S_i.$$

Let $P < \mathrm{SL}(W)$ be the corresponding *parabolic subgroup*, i.e. the stabilizer of the flag $\mathcal{F}$, and let $V < P$ be the associated *unipotent subgroup*, consisting of all $A \in P$ that act trivially on each of the factors W_i/W_{i-1}. We then have

$$P = V \rtimes R, \quad R = \left\{ (A_1, \ldots, A_n) \in \prod_{i=1}^n \mathrm{GL}(S_i) \;\middle|\; \prod_i \det A_i = 1 \right\}.$$

Elements of P can be considered as upper triangular $n \times n$-matrices, where the ij^{th} entry is an element of $\mathcal{L}(S_j, S_i)$, the vector space of linear maps $S_j \longrightarrow S_i$. Elements of R are diagonal matrices, and elements of V are upper unitriangular.

Take a subset $\mathcal{E} \subset \{1, \ldots, n\}^2$ consisting of pairs (i, j) with $i < j$ and such that if $(i, j) \in \mathcal{E}$, then $(i', j), (i, j') \in \mathcal{E}$ for $i' < i$ and $j' > j$. So, imagining elements of $\mathcal{E}$ as corresponding to matrix entries, we are considering subsets of entries above the diagonal, that are closed under 'going up' and 'going to the right'. Let $V_{\mathcal{E}} < P$ be the normal subgroup consisting of all matrices that are equal to the identity matrix except at entries corresponding to elements of $\mathcal{E}$, and let $\mathcal{K}_{\mathcal{E}} < R$ be the kernel of the R-action (by conjugation) on $V/V_{\mathcal{E}}$.

Theorem 1.3 (IRSs of parabolic subgroups)**.** *The ergodic IRSs of P are exactly the random subgroups of the form $V_{\mathcal{E}} \rtimes K$, where K is an ergodic IRS of $\mathcal{K}_{\mathcal{E}}$.*

The subgroups $V_{\mathcal{E}}$ above are exactly the normal subgroups of P that lie in V. So, a special case of the theorem is that an ergodic IRS of P that is contained in V is a normal subgroup of P. In fact, when proving Theorem 1.3, one first proves this special case, and then applies it to $H \cap V$ when H is a general ergodic IRS of P. Once one knows $H \cap V = V_{\mathcal{E}}$, the statement of Theorem 1.3 is not a surprise, since the only obvious way to construct an IRS H with $H \cap V = V_{\mathcal{E}}$ is to take a semidirect product with an IRS of $\mathcal{K}_{\mathcal{E}}$.

The group $\mathcal{K}_{\mathcal{E}}$ can be described explicitly via matrices. Let $\mathcal{I}$ be the set of all $i \in \{1, \dots, n\}$ such that if $i < n$, then $(i, i+1) \in \mathcal{E}$, and if $i > 1$, then $(i-1, i) \in \mathcal{E}$. Then $(A_1, \dots, A_n)$ acts trivially on $V/V_{\mathcal{E}}$ exactly when for each maximal interval $\{i, \dots, j\} \subset \{1, \dots, n\} \setminus \mathcal{I}$, there is some $\lambda \in \mathbb{R} \smallsetminus \{0\}$ such that $A_i = \cdots = A_j = \lambda I$. In a picture, if $\mathcal{E}$ consists of the starred entries below, then $(A_1, \dots, A_n) \in \mathcal{K}_{\mathcal{E}}$ can be any diagonal matrix with the diagonal entries below, subject to the additional condition $\prod_i \det A_i = 1$.

$$\begin{pmatrix} \lambda I & 0 & \star & \star & \star & \star & \star & \star \\ 0 & \lambda I & \star & \star & \star & \star & \star & \star \\ 0 & 0 & A_3 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \mu I & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \mu I & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \mu I & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & A_7 & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_8 \end{pmatrix} \tag{1}$$

This means that $\mathcal{K}_{\mathcal{E}}$ is isomorphic to the determinant 1 subgroup of a direct product of general linear groups. Note that the conjugation action of every element of R on $\mathcal{K}_{\mathcal{E}}$ is equal to a conjugation by an element of $\mathcal{K}_{\mathcal{E}}$, since R is generated by $\mathcal{K}_{\mathcal{E}}$ and its centralizer. So, every IRS of $\mathcal{K}_{\mathcal{E}}$ is an IRS of R.

1.3 Plan of the paper

The paper is organized as follows. In §2, we establish some preliminary results: we introduce in §2.1 a useful co-cycle associated to an IRS in $A \rtimes \Gamma$, prove two facts about finite measure preserving linear actions in §2.2, and prove the result about transverse IRSs in §2.3. Section 3 concerns IRSs of parabolic subgroups, and in §4 we prove Theorems 1.1 and 1.2.

1.4 Acknowledgments

We thanks the referee for a careful reading of the paper, a number of useful comments, and the suggestion to combine our work with [4] to give a proof of the Nevo–Stuck–Zimmer theorem for $\mathrm{SL}_n(\mathbb{Z})$, as described in Remark 1 above.

2 IRSs in general semidirect products

In this section we study semidirect products $G = A \rtimes \Gamma$, where Γ acts on A by automorphisms. As above, pr is the natural projection $G \to \Gamma$.

2.1 The cocycle S_H

Let H be a subgroup of G. For each $M \in \operatorname{pr} H$ let

$$S_H(M) = \{v \in A \,:\, (v, M) \in H\}.$$

Then $S_H(I) = H \cap A$ is a subgroup of A where $I \in \Gamma$ denotes the identity element.

Let $(v, M), (w, N) \in H$. Then $(v, M) \cdot (w, N) = (v \cdot Mw, MN) \in H$. It follows that

$$S_H(MN) = S_H(M) \cdot M S_H(N), \tag{2}$$

where multiplication here denotes that of sets: $B \cdot C = \{b \cdot c \,:\, b \in B, c \in C\}$.

Claim 2.1. *If $M \in \operatorname{pr} H$, then $S_H(M)$ is a left coset of $S_H(I)$.*

Here, Claim 2.1 and Equation (2) say that S_H is a *cocycle* $S_H : \operatorname{pr} H \longrightarrow S_H(I)\backslash A$.

Proof. Suppose (v, M) and (w, M) are elements of H. Then

$$H \ni (v, M) \cdot (w, M)^{-1} = (v, M) \cdot (M^{-1}w^{-1}, M^{-1}) = (v \cdot MM^{-1}w^{-1}, I) = (vw^{-1}, I).$$

And if (v, M) and (x, I) are elements of H, we have

$$H \ni (x, I) \cdot (v, M) = (x \cdot Iv, M) = (xv, M). \quad \square$$

We end this section with a useful observation. As we will apply it only when A is abelian, we use additive notation here. Let (w, N) be an arbitrary element of G, and let $(v, M) \in H$. Then $(v, M)^{(w,N)} = (N^{-1}v + N^{-1}(M - I)w, M^N) \in H^{(w,N)}$. (Here, $a^b = b^{-1}ab$.) Hence

$$S_{H^{(w,N)}}(M^N) = N^{-1} S_H(M) + N^{-1}(M - I)w. \tag{3}$$

2.2 Group actions preserving finite measures

Here are four useful lemmas.

Lemma 2.2. *Suppose that G is a locally compact second countable group, and the induced action of $Z \leq \mathrm{Aut}(G)$ on the space Sub_G preserves a finite measure μ that is supported on lattices. Then Z preserves the Haar measure of G.*

Proof. For some n, the set $\mathcal{S}$ of lattices with covolume in $[\frac{1}{n}, n]$ has positive measure. If Z does not preserve Haar measure ν, there is some $A \in Z$ with $A_*\nu = c\nu$ with $c > n^2$. The sets $A^i\mathcal{S}$, where $i \in \mathbb{Z}$, are then all disjoint and have the same positive measure. This is a contradiction. □

The three following lemmas are inspired by an argument Furstenberg used in his proof of the Borel density theorem [12, Lemma 3].

Lemma 2.3. *Suppose a group Z acts linearly on $\mathbb{R}^d$ preserving a finite measure m, and $V = \mathrm{Span}(\operatorname{supp} m)$. Then the image of the map $Z \longrightarrow \mathrm{GL}(V)$ is precompact.*

Proof. Restricting, it suffices to prove the lemma when $\mathrm{Span}(\operatorname{supp} m) = \mathbb{R}^d$. Let (z_n) be a sequence in Z. After passing to a subsequence, we can assume that there is some subspace $W \subset \mathbb{R}^d$ such that the maps $z_n|_W$ converge to some linear map $z : W \longrightarrow \mathbb{R}^d$, while $z_n(x) \to \infty$ if $x \in \mathbb{R}^d \setminus W$. For instance, one can take W to be any subspace that is maximal among those for which there exists a subsequence (z_{n_k}) with the property that $z_{n_k}(x)$ is bounded for all $x \in W$, and then pass to a subsequence of such a subsequence.

If in the above, we always have $W = \mathbb{R}^d$, we are done. So, assume $W \neq \mathbb{R}^d$. Pick a metric inducing the one-point compactification topology on $\mathbb{R}^d \cup \infty$ and let $D : \mathbb{R}^d \cup \infty \longrightarrow \mathbb{R}$ be the distance to the closed set $z(W) \cup \infty$. By the dominated convergence theorem,

$$\int D(x)\, dm(x) = \int D(z_n(x))\, dm(x) \longrightarrow 0,$$

so m is supported on $z(W)$. But as W is a proper subspace, so is $z(W)$. This contradicts our assumption that $\mathrm{Span}(\operatorname{supp} m) = \mathbb{R}^d$. □

Lemma 2.4. *Suppose that $\mathbb{R}^d = \oplus_i \mathcal{L}_i$, a direct sum of subspaces, and that μ is a finite Borel measure on the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$. Suppose that for each j, there is a linear map $A_j : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ that acts as a scalar map $v \mapsto \lambda_i v$ on each subspace $\mathcal{L}_i$, satisfies $\lambda_j > \lambda_i$ for $i \neq j$, and induces a map on the Grassmannian that preserves μ. Then μ is concentrated on subspaces $W \subset \mathbb{R}^d$ that are direct sums of subspaces of the $\mathcal{L}_i$:*

$$W = \oplus_i S_i, \quad S_i \subset \mathcal{L}_i.$$

Proof. The argument is similar to that of Lemma 2.3. Denote the Grassmannian of k-subspaces of $\mathbb{R}^d$ by $Gr(k,d)$, fix j and let $\mathcal{Z}_j$ be the closed subset of $Gr(k,d)$ consisting of all subspaces of the form $S_j \oplus P'$, where $S_j \subset \mathcal{L}_j$ and $P' \subset \oplus_{i \neq j} \mathcal{L}_i$. Given an element $P \in Gr(k,d)$, let $D_j(P)$ be the distance from P to $\mathcal{Z}_j$, with respect to some metric inducing the natural topology. Then for each $P \in Gr(k,d)$, we have $D((A_j)^n(P)) \to 0$ as $n \to \infty$. Hence, the dominated convergence theorem says that

$$\int D(P)\,d\mu(P) = \int D((A_j)^n(P))\,d\mu(P) \to 0.$$

So, μ is supported on $\mathcal{Z}_j$. This works for all j, so the lemma follows. □

Lemma 2.5. *Let V, W be two vector spaces and let $\mathcal{L}(V,W)$ be the space of all linear maps from V to W. Suppose that $X \subset \mathcal{L}(V,W)$ is a random subspace whose law is invariant under the action of $SL(V) \times SL(W)$. Then almost surely, X is either $\{0\}$ or $\mathcal{L}(V,W)$.*

Here, $(A,B) \in SL(V) \times SL(W)$ acts by sending $T \in \mathcal{L}(V,W)$ to ATB^{-1}.

Proof. $SL(V) \times SL(W)$ is semisimple, and hence is a *m.a.p. group*, in the sense of Furstenberg's paper [12]. By [12, Lemma 3] and the exterior power trick in the subsequent 'Theorem', any finite $SL(V) \times SL(W)$-invariant measure on the set of subspaces of $\mathcal{L}(V,W)$ is supported on subspaces that are invariant under the $SL(V) \times SL(W)$ action. But it is easy to check that the only such subspaces are the two trivial ones. □

Remark 2. The proof of Lemma 2.5 above is a bit silly since it relies on certain well-known facts, e.g. that semisimple groups are m.a.p., that are considerably harder to prove than Lemma 2.5 itself. Really, one can just prove the lemma by applying the arguments from Furstenberg's paper to certain well-chosen sequences of elements in $SL(V) \times SL(W)$. We encourage the reader to do this, while we lazily give the short proof above.

2.3 Transverse IRSs

Let A and Γ be locally compact, second countable topological groups, and suppose Γ acts by continuous automorphisms on A. Let Γ_{triv} be the kernel of the action, and let $G = A \rtimes \Gamma$ be the associated semidirect product.

We call a subgroup $H \leq G$ *transverse* if $H \cap A = \{0\}$. For example, in the direct product $A \times A$, the diagonal subgroup is transverse, as is the second factor.

Theorem 2.6 (Structure of transverse IRSs in semidirect products, part 1)**.** *Suppose $G = \mathbb{R}^d \rtimes \Gamma$ and H is a transverse IRS of $G = \mathbb{R}^d \rtimes \Gamma$. Then $\operatorname{pr} H \leq \Gamma_{triv}$ almost surely.*

Remark 3. Theorem 2.6 also applies when $G = S \rtimes \Gamma$ and S is a closed subgroup of $\mathbb{R}^d$. Indeed, the Γ-action on such an S extends to the span of S to which Theorem 2.6 applies, and any transverse IRS of $G = S \rtimes \Gamma$ induces a transverse IRS of $G = \mathrm{span(S)} \rtimes \Gamma$.

Remark 4. If the action $\Gamma \circlearrowleft A$ is faithful (as it is, for example, in the case of the special affine groups), then Theorem 2.6 implies there are no nontrivial transverse IRSs of G. Also, note that the theorem fails when A is not torsion-free abelian. For instance, if A is finite then a random conjugate of Γ is an IRS of $A \rtimes \Gamma$. And if A is not abelian, the antidiagonal

$$\{(g, g^{-1}) \mid g \in A\} \subset A \rtimes A,$$

where $a \in A$ acts on $x \in A$ by $a(x) = a^{-1}xa$, is a normal subgroup of $A \rtimes A$ that does not project into $A_{triv} = Z(A)$. However, we expect that for general A, if H is a transverse IRS of $A \rtimes \Gamma$, then the action of any element of $\operatorname{pr} H$ on A is well-approximated by inner automorphisms of A in some sense.

Proof of Theorem 2.6. Let H be a nontrivial transverse IRS of G. In order to get a contradiction, suppose that it is not the case that $\operatorname{pr} H \leq \Gamma_{triv}$ almost surely. Then there is an open subset $U \subset \Gamma$ with compact closure such that $U \cap \Gamma_{triv} = \emptyset$, and $\operatorname{pr} H \cap U \neq \emptyset$ with positive probability. In addition we choose U small enough so that for some $w \in \mathbb{R}^d$, some $0 < b_1 < b_2 \in \mathbb{R}_+$ and some linear $L : \mathbb{R}^d \longrightarrow \mathbb{R}$, we have that

$$b_1 \leq L((M - I)w) \leq b_2, \text{ for all } M \in U. \tag{4}$$

Choose a left Haar measure μ_H on $\operatorname{pr} H$. By [6, Claim A.2], this can be done so that the μ_H vary continuously with $H \in \mathrm{Sub}_G$, when regarded as measures on $\Gamma \geq \operatorname{pr} H$.

Because H is transverse, $S_H(M)$ is a single element of $\mathbb{R}^d$ for any $M \in \operatorname{pr} H$. Selecting first a random $H \in \mathrm{Sub}_G$ with $\operatorname{pr} H \cap U \neq \emptyset$, and then a μ_H-random $M \in \operatorname{pr} H \cap U$, we can interpret the cocycle $S_H(M)$ as an $\mathbb{R}^d$-valued random variable. Here, note that $\mu_H(\operatorname{pr} H \cap U)$ is always finite and nonzero, since $\operatorname{pr} H \cap U$ is nonempty, pre-compact and open in H.

Taking $w \in \mathbb{R}^d$ as in the first paragraph of the proof, let $H^w = (w, I)^{-1} H (w, I)$. Since $\operatorname{pr} H = \operatorname{pr} H^w$, we get a map $(H, M) \mapsto (H^w, M)$ defined on the domain

$$\big\{(H, M) \mid H \in \mathrm{Sub}_G,\ \operatorname{pr} H \cap U \neq \emptyset,\ M \in \operatorname{pr} H \cap U\big\} \tag{5}$$

of the random variable $S_H(M)$. As H is an IRS, this map is measure preserving, so the distributions of $S_{H^w}(M)$ and $S_H(M)$ are equal, say to a probability measure m_U on $\mathbb{R}^d$.

By (3), we have $S_{H^w}(M) = S_H(M) + (M - I)w$ for all $M \in \operatorname{pr} H = \operatorname{pr} H^w$. Iterating the conjugation by w and using (4),

$$L\big(S_H(M)\big) + nb_1 \leq L\big(S_{H^{nw}}(M)\big) \leq L\big(S_H(M)\big) + nb_2,\ \forall n \in \mathbb{N}. \tag{6}$$

This contradicts the fact that m_U is a probability measure. For suppose $[a_1, a_2] \subset \mathbb{R}$ is an interval with $m_U(L^{-1}([a_1, a_2])) > 0$. For a sufficiently sparse sequence $n_k \in \mathbb{N}$, the intervals $[a_1 + n_k b_1, a_2 + n_k b_2] \subset \mathbb{R}$ are all disjoint. Hence,

$$1 \geq \sum_k m_U\big(L^{-1}[a_1 + n_k b_1, a_2 + n_k b_2]\big) \geq \sum_k m_U(L^{-1}[a_1, a_2]) = \infty.$$

This contradiction proves the theorem. □

Theorem 2.7 (Structure of transverse IRSs in semidirect products, part 2). *Suppose $G = A \rtimes \Gamma$, A is a simply connected nilpotent Lie group, H is a transverse IRS of $G = A \rtimes \Gamma$ and λ is the law of H. Let*

$$\mathcal{H} = \cup_{H \in \operatorname{supp} \lambda} H.$$

If $\mathcal{V} \subseteq A$ is the Zariski closure of the set of first coordinates of all $(v, M) \in \mathcal{H}$, then $\mathcal{V}$ is Γ-invariant and the image of the map $Z(\operatorname{pr} \mathcal{H}) \longrightarrow \mathrm{Aut}(\mathcal{V})$ is precompact.

Here $Z(\operatorname{pr}\mathcal{H})$ denotes the centralizer of $\operatorname{pr}\mathcal{H}$ in Γ, and the Zariski closure of a subset of A is the smallest connected Lie subgroup of A containing that subset.

Remark 5. Theorem 2.7 also applies when $G = S \rtimes \Gamma$ and S is a closed subgroup of some simply connected nilpotent Lie group A. Indeed, the Γ-action on such an S extends to the Zariski closure $\overline{S}$ [17, Theorem 2.11], to which Theorem 2.7 applies, and any transverse IRS of $G = S \rtimes \Gamma$ induces a transverse IRS of $G = \overline{S} \rtimes \Gamma$. See [17, Chapter II] for more information about the 'Zariski closure' operation in simply connected nilpotent Lie groups, which behaves very similarly to 'span' in $\mathbb{R}^d$.

Remark 6. To illustrate Theorem 2.7, suppose $A = \Gamma = \mathbb{R}^2$ and $(s,t) \in \Gamma$ acts by a rotation on A with angle s. Then if

$$H_\theta = \Big\{ \big((t\cos\theta, t\sin\theta), (0,t) \big) \mid t \in \mathbb{R} \Big\} \leq A \rtimes \Gamma,$$

we obtain a transverse IRS of $G = A \rtimes \Gamma$ by randomly picking $\theta \in [0, 2\pi]$ against Lebesgue measure. Here, the centralizer $Z(\operatorname{pr}\mathcal{H})$ is all of Γ, which acts compactly on A.

Proof of Theorem 2.7. The Γ-invariance of $\mathcal{V}$ is immediate. For if $N \in \Gamma$ and $(v, M) \in \mathcal{H}$,

$$(e,N)^{-1}(v,M)(e,N) = (N^{-1}v, N^{-1}MN). \tag{7}$$

Here, we write e for the identity element since A is not necessarily abelian. As $\operatorname{supp}\lambda$ is conjugation invariant, the set of all $v \in A$ such that $(v, M) \in \mathcal{H}$ for some M is Γ-invariant. Hence, its Zariski closure $\mathcal{V}$ is also Γ-invariant.

As in the proof of Theorem 2.6, choose $U \subset \Gamma$ with compact closure such that $\operatorname{pr} H \cap U \neq \emptyset$ with positive probability. Let $N \in Z(\operatorname{pr}\mathcal{H})$ and write $H^N = (e,N)^{-1}H(e,N)$. Substituting $N^{-1}MN = M$ in (7) we see that $\operatorname{pr} H = \operatorname{pr} H^N$, so as before the distribution of $S_{H^N}(M)$ is the same as m_U, the distribution of $S_H(M)$. Now, though, (7) implies that

$$S_{H^N}(M) = N^{-1}(S_H(M)).$$

So, the measure m_U on A is $Z(\operatorname{pr}\mathcal{H})$-invariant.

Since A is a simply connected nilpotent Lie group, there is a diffeomorphism $\log : A \longrightarrow \mathfrak{a}$ to the Lie algebra $\mathfrak{a}$ that is an inverse for the Lie group exponential map [14, 1.127]. Then $\log_* m_U$ is a probability measure on $\mathfrak{a}$ that is invariant under the induced action of $Z(\operatorname{pr}\mathcal{H})$ on $\mathfrak{a}$. By Lemma 2.3, $Z(\operatorname{pr}\mathcal{H})$ acts precompactly on the span $V_U = \operatorname{Span}(\operatorname{supp}\log_* m_U)$, and therefore it acts precompactly on the sum V of all V_U, as U ranges over all possible choices. But the Zariski closure $\mathcal{V} = \exp(V)$, so then $Z(\operatorname{pr}\mathcal{H})$ acts precompactly on $\mathcal{V}$ as well. $\square$

We present an easy corollary of Theorem 2.6:

Corollary 2.8. *The only ergodic IRSs of the affine group $\mathbb{R} \rtimes \mathbb{R}^+$ are the point masses on its closed, normal subgroups: $\{e\}$, $\mathbb{R}$, $\mathbb{R} \rtimes \mathbb{R}^+$ and $\mathbb{R} \rtimes \{\alpha^n \mid n \in \mathbb{Z}\}$, where $\alpha > 0$.*

Note that this stands in contrast to other metabelian groups (e.g., lamplighter groups) that have a rich set of invariant random subgroups [9].

Proof of Corollary 2.8. Let H be a non-trivial ergodic IRS of $\mathbb{R} \rtimes \mathbb{R}^+$. If H is transverse, then $\operatorname{pr} H = \{1\} \in \mathbb{R}_+$, by Theorem 2.6. Hence $H = \{e\}$.

Otherwise, the random subgroup $H \cap \mathbb{R} \subset \mathbb{R}$ is nontrivial almost surely, and its law is invariant under the $\mathbb{R}^+$ action (i.e., multiplication by a scalar). So, $H \cap \mathbb{R} = \mathbb{R}$ almost surely, and $H = \mathbb{R} \rtimes \operatorname{pr} H$. But $\operatorname{pr} H$ is an ergodic IRS of $\mathbb{R}^+$, and thus must be a point mass on either $\{1\}$, $\mathbb{R}^+$ or $\mathbb{R} \rtimes \{\alpha^n \mid n \in \mathbb{Z}\}$, where $\alpha > 0$. We have thus proved the claim. $\square$

3 IRSs of parabolic subgroups

To recap our notation: $W = S_1 \oplus \cdots \oplus S_n$ is a real vector space, $\mathcal{F}$ is the associated flag

$$0 = W_0 < W_1 < \cdots < W_n = W, \quad W_k = \oplus_{i=1}^k S_i,$$

$P < \mathrm{SL}(W)$ is the parabolic subgroup stabilizing $\mathcal{F}$, $V < P$ is the unipotent subgroup of all $A \in P$ that act trivially on each of the factors W_i/W_{i-1}, and

$$P = V \rtimes R, \quad R = \left\{ (A_1, \ldots, A_n) \in \prod_{i=1}^n \mathrm{GL}(S_i) \;\middle|\; \prod_i \det A_i = 1 \right\}.$$

Also, $\mathcal{E} \subset \{1, \ldots, n\}^2$ will denote a subset of pairs (i, j) with $i < j$ that is closed under 'going up' and 'going to the right', and we will let $V_\mathcal{E} < P$ be the normal subgroup consisting of all matrices that are equal to the identity matrix except at entries corresponding to elements of $\mathcal{E}$. Let $\mathcal{K}_\mathcal{E} < R$ be the kernel of the R-action (by conjugation) on $V/V_\mathcal{E}$.

The goal of this section is to prove Theorem 1.3, i.e. that the ergodic IRSs of P are exactly the random subgroups of the form $V_\mathcal{E} \rtimes K$, where K is an ergodic IRS of $\mathcal{K}_\mathcal{E}$.

We start with the following lemma.

Lemma 3.1. *Suppose that H is an invariant random subgroup of P that lies in V. Then almost surely, $H = V_\mathcal{E}$ for some $\mathcal{E}$.*

Proof. Regard V as the space of upper unitriangular block matrices, where the ij^{th} entries is in $\mathcal{L}(S_j, S_i)$. It suffices to show that almost surely, H is a 'matrix entry subgroup', i.e. a subgroup determined by prescribing that some fixed subset of the matrix entries are all zero. As there are only finitely many such subgroups, it will follow that almost surely, H is a matrix entry subgroup of V that is a normal subgroup of P. A quick computation with elementary matrices shows that the only such subgroups are the $V_\mathcal{E}$ described above.

Let H_0 and $\overline{H}$ be the identity component and Zariski closure of H, respectively, recalling that the *Zariski closure* of a subgroup is the smallest connected Lie subgroup of V containing it. (See [17, Chapter II].) Then H_0 and $\overline{H}$ are both R-invariant random subgroups of V. Let $\mathfrak{h}_0$ and $\overline{\mathfrak{h}}$ be the associated Lie algebras, which are R-invariant random subspaces of the Lie algebra $\mathfrak{v}$ of V. One can identify $\mathfrak{v}$ with the set of all strictly upper triangular block matrices, where the ij^{th} entry is an element of $\mathcal{L}(S_j, S_i)$. If we identify $\mathcal{L}(S_j, S_i)$ with the subspace of $\mathfrak{v}$ consisting of matrices that are nonzero at most in the ij^{th} entry, then

$$\mathfrak{v} = \oplus_{i<j} \mathcal{L}(S_j, S_i).$$

The action $R \circlearrowright \mathfrak{v}$ leaves all the factors $\mathcal{L}(S_j, S_i)$ invariant. Given $k < l$, let A be the matrix that has a $2I$ in the kk^{th} entry and a $\frac{1}{2}I$ in the ll^{th} entry, and is otherwise equal to the identity matrix. Then the matrix $\frac{1}{\det A}A$ lies in R, and acts by conjugation on each $\mathcal{L}(S_j, S_i)$ as the scalar matrix $\frac{\lambda_{ij}}{\det A}I$, where

$$\lambda_{ij} = \begin{cases} 4 & (i,j) = (k,l) \\ 2 & i = k, j \neq l \text{ or } j = l, i \neq k \\ \frac{1}{2} & i = l \text{ or } j = k, \text{ and } i \neq j \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

Applying Lemma 2.4 to the direct sum $\mathfrak{v} = \oplus_{i<j}\mathcal{L}(S_j, S_i)$, considered together with the actions of all the matrices A obtained by varying k, l, we see that almost surely, both $\mathfrak{h}_0$ and $\overline{\mathfrak{h}}$ are direct sums of subspaces of the factors $\mathcal{L}(S_j, S_i)$. However, the only R-invariant random subspaces of a *fixed* factor $\mathcal{L}(S_j, S_i)$ are the zero subspace and the entire $\mathcal{L}(S_j, S_i)$. (This follows immediately from Lemma 2.5, since one can embed $SL(S_i) \times SL(S_j) \hookrightarrow R$ by taking (A, B) to the element of R that has $A \in \mathcal{L}(S_i, S_i)$ in the ii entry and $B \in \mathcal{L}(S_j, S_j)$ in the jj entry, and is otherwise equal to the identity matrix.) Hence, $\mathfrak{h}_0$ and $\overline{\mathfrak{h}}$ are almost always direct sums of the factors $\mathcal{L}(S_j, S_i)$ themselves, rather than subspaces thereof. In other words, H_0 and $\overline{H}$ are matrix entry subgroups almost surely.

Now $H_0 \subset H \subset \overline{H}$, so if $H_0 = \overline{H}$, then H is a matrix entry subgroup as desired. So, after restricting the law of H, we may assume that almost surely H_0 and $\overline{H}$ are *fixed* matrix entry subgroups and that $H_0 \subsetneq H$. As H is an IRS of P, H_0 is a normal subgroup of P. We can then project H to a P-invariant random subgroup H/H_0 of the quotient group $\overline{H}/H_0$. Since V is a nilpotent Lie group, the sub-quotient group $\overline{H}/H_0$ is as well. Every Zariski dense subgroup of a nilpotent Lie group is a lattice (c.f. [17, Theorem 2.3]), so the P-invariant random subgroup $H/H_0 < \overline{H}/H_0$ is a lattice almost surely. Lemma 2.2 then implies that the P action on $\overline{H}/H_0$ preserves Haar measure.

But if $\mathcal{D}$ is the set of matrix entries that in $\overline{H}$ are free to take on any value, and in H_0 are prescribed to be zero, there is a diffeomorphism

$$\overline{H}/H_0 \longrightarrow \oplus_{(i,j)\in\mathcal{D}}\mathcal{L}(S_j, S_i)$$

that takes a matrix in $\overline{H}$ to the list of its $\mathcal{D}$-entries. If Lebesgue measures are chosen on the Euclidean spaces $\mathcal{L}(S_j, S_i)$, the resulting product measure pulls back to a Haar measure on $\overline{H}/H_0$. So, one can witness that the action $R \circlearrowright \overline{H}/H_0$ does not preserve Haar measure as follows. Let $i_{\min}$ be the minimum i such that there is some $(i, j) \in \mathcal{D}$, and $i_{\max}$ be the maximum i such that there is some $(j, i) \in \mathcal{D}$, and define $A \in R$ by letting

$$A_{ii} = \begin{cases} 2^{1/\dim(S_{i_{min}})}I & i = i_{\min} \\ 2^{-1/\dim(S_{i_{max}})}I & i = i_{\max} \\ I & \text{otherwise.} \end{cases}$$

This A acts diagonally on $\oplus_{(i,j)\in\mathcal{D}}\mathcal{L}(S_i, S_j)$, and the action is scalar in each factor. Moreover, there are no entries of $\mathcal{D}$ directly above the $i_{\min}$ diagonal entry, and no entries to the right of the $i_{\max}$ diagonal entry, so the eigenvalues of the action of A on $\oplus_{(i,j)\in\mathcal{D}}\mathcal{L}(S_i,S_j)$ are 1, $2^{1/\dim(S_{i_{min}})}$ an $2^{1/\dim(S_{i_{max}})}$. Hence, A cannot preserve Lebesgue measure. $\square$

Now suppose that H is an ergodic IRS of $P = V \rtimes R$. Lemma 3.1 implies that there is some $\mathcal{E}$ such that $H \cap V = V_{\mathcal{E}}$ almost surely. Applying Theorem 2.6 to the transverse IRS that is the projection of H to $(V/V_{\mathcal{E}})^{ab} \rtimes R$, where $(\,\cdot\,)^{ab}$ is abelianization, we see that $\mathrm{pr}\, H \subset R$ almost surely acts trivially on $(V/V_{\mathcal{E}})^{ab}$. But if $\mathcal{A}$ is the set of super diagonal entries in our block matrices that do not lie in $\mathcal{E}$, there is an isomorphism

$$(V/V_{\mathcal{E}})^{ab} \longrightarrow \oplus_{(i,j)\in\mathcal{A}}\mathcal{L}(S_i,S_j)$$

that comes from taking a matrix in V to its list of $\mathcal{A}$-entries. It follows that a matrix in R acts trivially on $(V/V_{\mathcal{E}})^{ab}$ if and only if it acts trivially on $V/V_{\mathcal{E}}$: triviality of the $(V/V_{\mathcal{E}})^{ab}$-action is enough to force the conditions on diagonal entries indicated in the matrix (1) from the introduction. Hence, $\mathrm{pr}\, H$ almost surely lies in the kernel $\mathcal{K}_{\mathcal{E}}$ of the $V/V_{\mathcal{E}}$-action as desired.

We now know that $H \cap V = V_{\mathcal{E}}$ and $\mathrm{pr}\, H \subset \mathcal{K}_{\mathcal{E}}$ almost surely. We would like to conclude that H has the form $V_{\mathcal{E}} \rtimes K$ for some IRS $K < K_{\mathcal{E}}$. Note that this is not immediately obvious—the diagonal in $\mathbb{R}^2$ is a normal subgroup that intersects the first factor trivially, but does not split as a product of subgroups of the two factors. By Theorem 2.7, we know that the centralizer $Z(\mathrm{pr}\, H) \subset R$ acts precompactly on $\mathcal{X} \subset V/V_{\mathcal{E}}$, where $\mathcal{X}$ is the Zariski closure in $V/V_{\mathcal{E}}$ of the projections of all first coordinates of elements $(v, M) \in H$. If $\mathcal{X} = \{V_{\mathcal{E}}\}$, we are done, since then the first coordinates of all $(v, M) \in H$ lie in $V_{\mathcal{E}} = H \cap V$ and H must have the form $V_{\mathcal{E}} \rtimes K$ for some IRS $K < \mathcal{K}_{\mathcal{E}}$.

So, we may assume that $\mathcal{X}\mathcal{V}_{\mathcal{E}} \supsetneq V_{\mathcal{E}}$. Picking a matrix B in the difference, there is some entry $(i,j) \notin \mathcal{E}$ in which B is nonzero. The centralizer $Z(\mathrm{pr}\, H)$ contains all elements of R all of whose diagonal entries are scalars, so in particular it contains the matrix whose eigenvalues λ are listed in (8) above. The action of this matrix on B scales the (i,j) entry by 4, so $Z(\mathrm{pr}\, H)$ does not act pre-compactly on $\mathcal{X}$, and we have a contradiction.

4 IRSs of special affine groups

Using Theorems 2.6 and 2.7, it is now fairly easy to prove the results on IRSs of special affine groups stated in the introduction.

Proof of Theorem 1.1. Let H be a nontrivial ergodic IRS of $\mathbb{R}^d \rtimes \mathrm{SL}_d(\mathbb{R})$. Suppose that $H \cap \mathbb{R}^d = \{0\}$ almost surely. As the action $\mathrm{SL}_d(\mathbb{R}) \circlearrowright \mathbb{R}^d$ is faithful, Theorem 2.6 implies that H is trivial. So, $H \cap \mathbb{R}^d$ is almost surely some nontrivial subgroup of $\mathbb{R}^d$.

In order to prove $H \cap \mathbb{R}^d$ is either a lattice or $\mathbb{R}^d$, it suffices to prove that the Zariski closure of $H \cap \mathbb{R}^d$ is almost surely $\mathbb{R}^d$. If not, we get for some $1 \le k \le d-1$, a $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure on the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$. In the terminology of Furstenberg [12], $\mathrm{SL}_d(\mathbb{R})$ is a m.a.p. group, so this measure must be concentrated on $\mathrm{SL}_d(\mathbb{R})$-invariant points. (Apply [12, Lemma 3] to the k^{th} exterior power of $\mathbb{R}^d$.) However, no nontrivial subspaces of $\mathbb{R}^d$ are $\mathrm{SL}_d(\mathbb{R})$-invariant.

Now suppose $H \cap \mathbb{R}^d$ is a lattice (almost surely). Let μ denote the law of H. By decomposing μ over the map $H \mapsto H \cap \mathbb{R}^d$, we can write $\mu = \int \mu_\Lambda \, d\nu(\Lambda)$ where ν is the pushforward of μ under $H \mapsto H \cap \mathbb{R}^d$ and μ_Λ is concentrated on the set of subgroups H such that $H \cap \mathbb{R}^d = \Lambda$. By ergodicity ν is supported on the set of lattices of some fixed covolume $c > 0$. Moreover ν is $\mathrm{SL}_d(\mathbb{R})$ invariant since the map $H \mapsto H \cap \mathbb{R}^d$ is equivariant. Since $\mathrm{SL}_d(\mathbb{R})$ acts transitively on this set of lattices, it follows that ν must be the Haar measure.

By equivariance, we must have $\mu_{g\Lambda} = g_* \mu_\Lambda$ for $g \in \mathrm{SL}_d(\mathbb{R})$ and ν-a.e. Λ. Because $\mathrm{SL}_d(\mathbb{R})$ acts transitively on the set of lattices with fixed covolume, we can assume without loss of generality that $\mu_{g\Lambda} = g_* \mu_\Lambda$ holds for every $g \in \mathrm{SL}_d(\mathbb{R})$ and lattice Λ.

We claim that μ_Λ-a.e. H is contained in $\Lambda \rtimes \mathrm{SL}(\Lambda)$. First let $(v, M) \in H$. For any $w \in \Lambda$ we have that $(w, I) \in H$, and so

$$(v, M)(w, I)(v, M)^{-1} = (Mw, I) \in H \cap \mathbb{R}^d = \Lambda.$$

Because $w \in \Lambda$ is arbitrary, $M \in \mathrm{SL}(\Lambda)$. Next observe that the law of H is invariant under conjugation by $\Lambda \rtimes \mathrm{SL}(\Lambda)$. So if there exists $M \in \mathrm{SL}(\Lambda)$ such that $S_H(M) \neq \Lambda$ with positive probability then $MHM^{-1} \cap \mathbb{R}^d \neq \Lambda$ with positive probability. This contradiction shows that $S_H(M) = \Lambda$ almost surely which implies $H \leq \Lambda \rtimes \mathrm{SL}(\Lambda)$. Thus μ_Λ is the law of an IRS of $\Lambda \rtimes \mathrm{SL}(\Lambda)$. This IRS must be ergodic because μ is ergodic.

□

Proof of Theorem 1.2. Let H be a non-trivial, ergodic IRS of $G = \mathbb{Z}^d \rtimes \mathrm{SL}_d(\mathbb{Z})$. Then $H \cap \mathbb{Z}^d$ is a random subgroup of $\mathbb{Z}^d$ whose law is invariant to the $\mathrm{SL}_d(\mathbb{Z})$ action. Note that since the action $\mathrm{SL}_d(\mathbb{Z}) \circlearrowright \mathbb{Z}^d$ is faithful, Theorem 2.6 implies that $H \cap \mathbb{Z}^d \neq \{0\}$. Since there are only countably many subgroups of $\mathbb{Z}^d$, the distribution of $H \cap \mathbb{Z}^d$ must be concentrated on a single, finite $\mathrm{SL}_d(\mathbb{Z})$-orbit. So, $H \cap \mathbb{Z}^d$ is almost surely finite index in $\mathbb{Z}^d$.

Let $O = \{M(H \cap \mathbb{Z}^d) \, : \, M \in \mathrm{SL}_d(\mathbb{Z})\}$ be the orbit of $H \cap \mathbb{Z}^d$ under the $\mathrm{SL}_d(\mathbb{Z})$ action. Now, the intersection of the groups in this orbit is also finite index in $\mathbb{Z}^d$, and is furthermore $\mathrm{SL}_d(\mathbb{Z})$-invariant, and so must equal $n\mathbb{Z}^d$ for some $n \in \mathbb{N}$.

Recall that $G_n = (n\mathbb{Z}^d) \rtimes \Gamma(n)$, and let $H_n = H \cap G_n$, a finite index subgroup of H. Using the cocycle notation of §2.1, for any $M \in \mathrm{pr}\, H_n$ it holds that $S_H(M) = S_H(I) := H \cap \mathbb{Z}^d$, since otherwise $S_H(M)$ is a non-trivial coset of $S_H(I)$, and its intersection with $n\mathbb{Z}^d$, a subgroup of $S_H(I)$, is trivial, thus excluding M from $\mathrm{pr}\, H_n$. It follows that $H_n = (n\mathbb{Z}^d) \rtimes (\mathrm{pr}\, H_n)$. This completes the proof of Theorem 1.2. □

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