TRANSITIVE GRAPHS UNIQUELY DETERMINED BY THEIR LOCAL STRUCTURE

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Abstract. We give an example of an infinite, vertex transitive graph that has the following property: it is the unique completion to a transitive graph of a large enough finite subgraph of itself.

1. Introduction

Vertex transitive graphs "look the same from the point of view of every vertex"; all vertices play the same role in their geometry. Thus they are a natural model for a discrete, homogeneous geometrical space. In this paper we study transitive graphs whose local structure determines their global structure.

Consider the following scenario: Alice has in mind some vertex transitive graph G and wants to describe it to Bob. Her graph may be infinite, and so she cannot provide a complete list of the vertices and edges. Instead, she chooses a vertex in the graph, and shows to Bob a ball of some finite radius around that vertex; since the graph is transitive, it does not matter which vertex she chooses. Can this convey to Bob enough information to uniquely determine G, given that he knows that G is transitive?

For example, if G is the bi-infinite chain, then the answer is no: a ball of any radius is a finite chain, and so Bob cannot tell whether G is the bi-infinite chain, or whether it is a large cycle. If G is a regular tree, then likewise the answer is no: there are many transitive graphs that locally look like trees, but are not trees. On the other hand, if G is finite then the answer is yes, since in this case Alice can show Bob the entire graph. Hence the question is: does there exist an infinite transitive graph that is uniquely determined by a large enough finite ball?

Formally, let $\mathcal{G}$ be the set of finite or countably infinite, simple, undirected, locally finite, connected, vertex transitive graphs; these terms are defined formally in Section 2. Given a $G \in \mathcal{G}$ and an $r \in \mathbb{N}$, a ball of radius r in G is the subgraph that includes all vertices at

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distance at most r from some vertex in G, and all edges between them. We say that $G = (V, E) \in \mathcal{G}$ is *isolated* if it has the following property: there exists an $r \in \mathbb{N}$ large enough so that, if a ball of radius r in some $H \in \mathcal{G}$ is isomorphic to the ball of radius r in G, then H is isomorphic to G. Intuitively, the structure of the ball of radius r in G determines G uniquely.

Clearly, every finite transitive graph is isolated: one can take r to be the radius of G. However, it is not obvious that there are any *infinite* graphs that have this property. In this paper we give an example of an isolated infinite graph, namely Trofimov's *grandfather graph* [9].

Note that the grandfather graph is not unimodular, and so cannot locally resemble finite graphs. The novelty is therefore that it can also not locally resemble any other infinite graph. It would be interesting to find an example of an isolated, finite, unimodular graph.

This question can be formulated as one of finding isolated points in a natural topology on the set of transitive graphs, namely the Benjamini-Schramm topology [2, 4]. This topological perspective raises a number of interesting questions: what is the Cantor-Bendixson rank of this space? Which graphs are left after the isolated points are repeatedly removed? And what generic properties do these graphs have?

These and similar questions have been previously addressed in regard to the related space of *marked groups* [5, 6]. In particular, Cornulier, Guyot and Pitsch [7] characterize the isolated points in that space. It would be interesting to understand if the (unlabeled) Cayley graphs of these groups are isolated in the space of transitive graphs.

An analogous, more quantitative version of this question can be asked for finite graphs: which finite transitive graphs of radius n are uniquely determined by a ball of size (say) $n/10$?

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2. Formal definitions and results

2.1. **Transitive graphs.** Let $G = (V, E)$ be a graph. We will study the set of graphs with the following properties:

- V is finite or countably infinite.
- G is simple and undirected: E is a symmetric relation on V.
- G is locally finite: the number of edges incident on each vertex is finite.
- G is connected: there is a path between every pair of vertices.
- G is vertex transitive; we next define this notion.

A *graph isomorphism* between $G = (V, E)$ and $H = (U, F)$ is a bijection $h\colon V \to U$ such that $(u, w) \in E$ if and only if $(h(u), h(w)) \in F$. A graph automorphism is a graph isomorphism from a graph to itself. A graph $G = (V, E)$ is said to be *vertex transitive* if its automorphism group acts transitively on its vertices. That is, if for every $u, w \in V$ there exists an automorphism h such that $h(u) = w$. The *isomorphism class* of a transitive graph G is the set of graphs H that are isomorphic to G. We denote by $\mathcal{G}$ the set of isomorphism classes of graphs with the properties described above. In this paper, we will, whenever unambiguous, refer to "graph isomorphism classes" simply as "graphs", and likewise simply denote by G the isomorphism class of G. We will accordingly write $G = H$ whenever G and H are in the same isomorphism class.

Given $G = (V, E) \in \mathcal{G}$ and $r \in \mathbb{N}$, let $B_r(G) = (V_r, E_r)$ be the ball of radius r in G. This is the finite induced subgraph of G whose vertices V_r are all the vertices at distance at most r from some vertex of G, and whose edges E_r are the edges of G whose vertices are both in V_r. Since we are concerned with graph isomorphism classes, and since G is vertex transitive, it does not matter with which vertex of G we choose to construct $B_r(G)$.

2.2. **The Benjamini-Schramm topology and isolated points.**

The Benjamini-Schramm topology [2,4] on $\mathcal{G}$ is defined by the following metric. Given $G, H \in \mathcal{G}$, let

$$D(G, H) = \sup\{2^{-r} \,:\, B_r(G) = B_r(H)\}.$$

It is straightforward to verify that this is indeed a metric. In fact, this topology is Polish and zero-dimensional. The sets $\mathcal{G}_d$ consisting of the graphs with degree d are compact in this topology.

We say that $G \in \mathcal{G}$ is *isolated* if it is an isolated point in this topology. By the above definition, this means that there exists an $r \in \mathbb{N}$ such that whenever $B_r(G) = B_r(H)$ then $G = H$. Since $B_r(G) = G$ for every finite G and r large enough, it follows immediately that all the finite graphs are isolated.

2.3. **The grandfather graph.**

The grandfather graph of order $n \geq 3$, G_n, is the following graph (see Figure 1). Let $\mathbb{T}_n$ be the regular tree of degree n. The ends of $\mathbb{T}_n$ can be identified with the set of infinite simple paths starting at o, an arbitrary distinguished vertex. Choose a distinguished end. Then each vertex has a unique edge in the direction of this end. Call the vertex on the other side of that edge the "father". Then each vertex has a unique father, and, as one can imagine, each vertex has a unique "grandfather". The set of vertices of G_n is identical

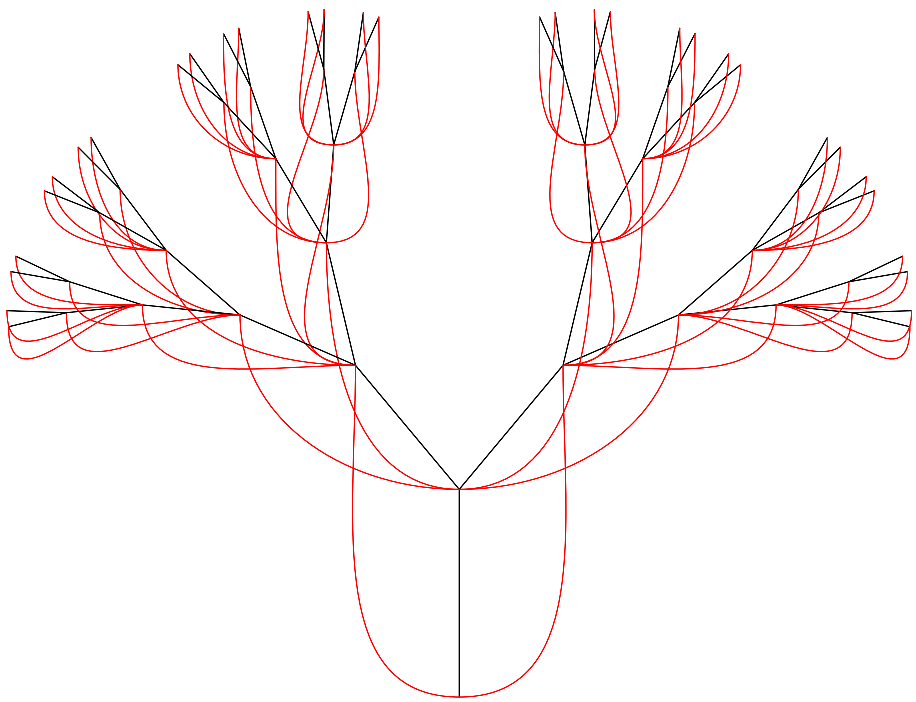

FIGURE 1. The grandfather graph G_3. Edges of $\mathbb{T}_3$ are straight black lines. Edges to grandfathers are red curves. The distinguished end is the "down" direction.

to that of $\mathbb{T}_n$. The set of edges includes the set of edges of $\mathbb{T}_n$, and in addition an edge between each vertex and its grandfather.

2.4. **Main result.**

Theorem 1. *For $n \geq 3$, the grandfather graph G_n is isolated.*

That is, there exists an $r > 0$ such that G_n has a unique ball of radius r among all vertex transitive graphs. In fact, we show that this already holds for $r = 1$.

We state here without proof that this result can be further extended to some classes of graphs that are similar to G_n. For example, the product of G_n with any finite graph will also be isolated, as will "great^k-grandfather" graphs.

3. MORE ON THE GRANDFATHER GRAPH

A *directed edge* in an undirected graph $G = (V, E)$ is an ordered pair (u, w) of vertices in G such that $(u, w) \in E$.

Let (u, w) and (u', w') be two directed edges in a graph G. We say that (u, w) and (u', w') are isomorphic if there exists a graph isomorphism of G that maps u to u' and w to w' (compare to the notion of "doubly rooted graphs" - see, e.g., [1, 8]). While all vertices in a transitive graph are isomorphic, not all directed edges are necessarily isomorphic.

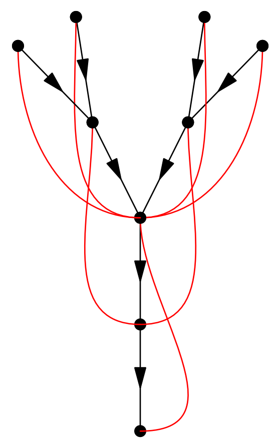

FIGURE 2. The ball of radius one in the grandfather graph G_3. The directions and labels of the edges can be inferred from the undirected graph.

In the grandfather graph G_n, (u, w) and (u', w') are isomorphic if and only if both pairs can be described by the same (ordered) familial relation: that is, if w is u's father (respectively son / grandfather / grandson) and w' is u''s father (respectively son / grandfather / grandson). This is a well-known property of this graph that is related to the fact that it is not unimodular (see, e.g., [3, 8]). In fact, one can already infer the familial relations by examining the ball of radius one: in this subgraph (see Figure 2), the father and the sons can be distinguished from the grandfather and the grandsons, since father-son pairs have n common neighbors, while grandfather-grandson pairs have only one. Furthermore, a node's father can be distinguished from the sons, since the father is connected to all of the sons (he is their grandfather), whereas the sons are not connected to each other. Thus, if w is u's father but w' is not u''s father, there is no graph isomorphism of G_n that maps (u, w) to (u', w').

We can therefore label each directed edge as a father / son / grandfather / grandson edge (that is, (u, w) will be a father edge if w is u's father), and this labeling will be invariant to any isomorphism of the graph.

This labeling gives rise to an equivalent definition of the grandfather graph: define a father relation on the vertices of n-regular tree $\mathbb{T}_n$; this is any relation in which each node has a unique father which is its neighbor in the graph. Then, connect each node to its grandfather.

The choice of a father relation is equivalent to a choice of end, and hence this also results in the grandfather graph.

4. Proof of Theorem 1

Let $H = (V, E)$ be any graph in $\mathcal{G}$ such that $B_1(H) = B_1(G_n)$. We will prove the theorem by showing that it is isomorphic to G_n. Note that Figure 2, depicts $B_1(G_3)$ and hence also $B_1(H)$, for the case $n = 3$.

Consider any two neighboring vertices u and w in H. Since $B_1(H) = B_1(G_n)$, u and w will either have one common neighbor or n common neighbors (see Figure 2). In the first case, color the edge (u, w) red; in the grandfather graph this will occur when one is the grandfather of the other. In the second case color the edge (u, w) black; in the grandfather graph this will occur when one is the father of the other.

We next would like to determine the direction of the black (father-son) edges; that is, we would like to know who is the son and who is the father. Fix two vertices u and w that are connected by a black edge. Since $B_1(H) = B_1(G_n)$, it will either be that case that (1) there is a unique path from u to w that first traverses a black edge and then a red edge or conversely (2) there is a unique path from u to w that first traverses a black edge and then a red edge (see Figure 2 again). In the first case we say that u is w's father, and in the second case w is u's father. Note that exactly one of these two cases must occur, and that indeed each vertex will have a unique father and $n - 1$ sons. We will call the directed edge (u, w) a father (resp., son) edge if w is u's father (resp., son).

Note that the resulting father relation on the vertices of H is invariant to the isomorphism group of H. We will use this to show that H is isomorphic to G_n, which will prove Theorem 1.

A simple cycle in a graph is a sequence of directed edges $(u_0, w_0), \ldots, (u_{k-1}, w_{k-1})$ such that $w_i = u_{i+1 \mod k}$, and each edge is visited at most once.

Claim 4.1. *There are no simple cycles in H which include only father edges and son edges.*

Proof. Assume by contradiction that $(u_0, w_0), \ldots, (u_{k-1}, w_{k-1})$ is a simple cycle comprised only of father edges and and son edges. Then all edges are of the same type (i.e., all father edges or all son edges): otherwise, there must be in the cycle a father edge followed by a son edge, which would make the cycle non-simple, since fathers are unique.

By perhaps changing the direction of the cycle we can therefore assume without loss of generality that all edges are father edges. Now, every node in H has a unique father and exactly $n - 1$ sons. Hence each node on the cycle is its own k^{th}-order father, and each node has

$n - 2 > 0$ sons which are not on the cycle. Since the father relation is invariant to graph isomorphisms, so is the k^{th}-order father relation.

Let u be a vertex on the cycle, and let v be vertex which is not on the cycle and is a son of u. Then there is no graph isomorphism of H that maps v to u, since v - unlike u - is not its own k^{th}-order father. Hence H is not transitive, and we have reached a contradiction. □

Remark. This claim can also be proved by showing that H is not unimodular and analyzing the Haar measure of the stabilizers of the nodes lying on the cycle (see [8]).

It follows from Claim 4.1 that the restriction of H to father-son edges is isomorphic to $\mathbb{T}_n$, the n-regular tree. This restriction is still a connected graph, since grandfather-grandson edges only connect nodes already connected by length two paths of father-son edges.

Since $B_1(H) = B_1(G_n)$, the grandfather edges in H are determined by the father-son relation, and in the same way that they are determined in G_n. Hence H can be constructed by adding grandfather edges to $\mathbb{T}_n$, equipped with a father relation. It follows that H is isomorphic to G_n, thus proving Theorem 1.

References

[1] David Aldous and Russell Lyons, *Processes on unimodular random networks*, Electronic Journal of Probability **12** (2007), no. 54, 1454–1508. MR2354165 (2008m:60012)

[2] David Aldous and J. Michael Steele, *The objective method: Probabilistic combinatorial optimization and local weak convergence*, Probability on Discrete Structures (Volume 110 of Encyclopaedia of Mathematical Sciences), ed. H. Kesten **110** (2003), 1–72.

[3] Itai Benjamini and Nicolas Curien, *Ergodic theory on stationary random graphs*, Electron. J. Probab **17** (2012), no. 93, 1–20.

[4] Itai Benjamini and Oded Schramm, *Recurrence of distributional limits of finite planar graphs*, Electronic Journal of Probability (2001), 1–13.

[5] Christophe Champetier, *L'espace des groupes de type fini*, Topology **39** (2000), no. 4, 657–680.

[6] Yves Cornulier, *On the cantor-bendixson rank of metabelian groups*, arXiv preprint arXiv:0904.4230 (2009).

[7] Yves Cornulier, Luc Guyot, and Wolfgang Pitsch, *On the isolated points in the space of groups*, Journal of Algebra **307** (2007), no. 1, 254–277.

[8] Vladimir Trofimov, *Automorphism groups of graphs as topological groups*, Matematicheskie Zametki **38** (1985), no. 3, 378–385.

[9] Russell Lyons with Yuval Peres, *Probability on tress and networks*, Cambridge University Press, 2015. In preparation. Current version available at `http://pages.iu.edu/~rdlyons/`.

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