HOMOMORPHISMS TO $\mathbb{R}$ OF AUTOMORPHISM GROUPS OF ZERO ENTROPY SHIFTS

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Abstract. We show that the automorphism group of every zero entropy infinite shift admits a "drift" homomorphism to $(\mathbb{R}, +)$ that maps the shift map to 1. This homomorphism arises as the expectation, under an invariant measure, of a cocycle defined on a space of asymptotic pairs.

1. Introduction

Automorphism groups of low complexity shifts have attracted much attention in the past few years (see, e.g., [5–9, 16, 17]), and this paper builds on this work. We recall the basic definitions and state our main result.

Let A be a finite set called an *alphabet*. The set $A^{\mathbb{Z}}$, endowed with the product topology, is called the *full shift*. Let $\sigma\colon A^{\mathbb{Z}} \to A^{Z}$ denote the *shift map*, given by $[\sigma(x)]_n = x_{n-1}$. A closed, σ-invariant subset of $A^{\mathbb{Z}}$ is called a *shift*. An infinite shift can be either countable or uncountable.

Let $\Sigma \subseteq A^{\mathbb{Z}}$ be a shift. A *word* of length n in Σ is an element $w \in A^n$ such that $w = (x_1, \ldots, x_n)$ for some $x \in \Sigma$. The number of words of length n in Σ is denoted by $P_\Sigma(n)$. The entropy of Σ is given by

$$h(\Sigma) = \lim_n \frac{1}{n} \log P_\Sigma(n).$$

The automorphism group of a shift Σ, denoted $\mathrm{Aut}(\Sigma)$, is the group of homeomorphisms of Σ that commute with σ. Note that σ (or, more precisely, its restriction to Σ) is an element of $\mathrm{Aut}(\Sigma)$.

Our main result shows that when Σ is zero entropy and infinite, then $\mathrm{Aut}(\Sigma)$ is *indicable*: it admits a non-trivial homomorphism to the additive group $(\mathbb{R}, +)$.

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Theorem 1. *Let Σ be a zero entropy infinite shift. Then there exists a group homomorphism $\Phi\colon \mathrm{Aut}(\Sigma) \to \mathbb{R}$ such that $\Phi(\sigma) = 1$.*

We construct Φ by defining a action of $\mathrm{Aut}(\Sigma)$ by homeomorphisms on a space $CA(\Sigma)$ of *asymptotic pairs*. We show that this space admits a bounded "drift" cocycle $c\colon \mathrm{Aut}(\Sigma) \times CA(\Sigma) \to \mathbb{Z}$ that satisfies $c(\sigma, \cdot) = 1$. Furthermore, using a technique introduced in [11], we show that $CA(\Sigma)$ also admits an $\mathrm{Aut}(\Sigma)$-invariant probability measure ν. The drift homomorphism Φ is defined as the expectation of c with respect to ν: $\Phi(\varphi) = \int c(\varphi, \cdot)\, \mathrm{d}\nu$.

This homomorphism has a similar flavor to those that stem from Krieger's dimension representation [4, 15]. The construction of a homomorphism through the integration of a cocycle with respect to an invariant measure is a technique that has yielded other interesting results in the past (see, e.g., Karlsson and Ledrappier [13]).

The remainder of this paper contains definitions and a proof of Theorem 1. In §7 we provide some examples and further notes.

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2. The space of calibrated asymptotic pairs

As is well known (see, e.g., [1, Chapter 2]) every infinite shift Σ admits at least one *asymptotic pair*: $x, y \in \Sigma$ such that $x_M \neq y_M$ for some $M \in \mathbb{Z}$, and $x_n = y_n$ for all $n < M$. Accordingly, given an asymptotic pair, we denote by

$$M(x, y) = \min\{m \in \mathbb{Z} \ : x_m \neq y_m\}$$

the first coordinate in which x and y differ. Note that

$$M(\sigma^k x, \sigma^k y) = M(x, y) + k. \tag{2.1}$$

Asymptotic pairs have been used to study automorphism groups of shifts: in [9] it is shown that $\mathrm{Aut}(\Sigma)$ is virtually $\mathbb{Z}$ if Σ is transitive and $\liminf_n P_\Sigma(n)/n$ is finite.

We say that an asymptotic pair is *calibrated* if $M(x, y) = 0$. If (x, y) is an asymptotic pair then $(\sigma^m(x), \sigma^m(y))$ is an asymptotic pair, and it is calibrated if and only if $m = -M(x, y)$. We denote by

$$C(x, y) = (\sigma^{-M(x,y)}x, \sigma^{-M(x,y)}y)$$

the calibrated asymptotic pair that is attained from (x, y) by shifting both of them so that they first differ at 0. We denote by $CA(\Sigma)$ the set of calibrated asymptotic pairs in Σ. This definition is closely related to the *asymptotic components* of [10] and the *asymptotic composants* of [3]. It is straightforward to see that $CA(\Sigma)$ is a closed subset of Σ^2, and is therefore compact. Note also that $C(x, y) \in CA(\Sigma)$ for every asymptotic pair (x, y).

3. An Aut(Σ) action on the calibrated asymptotic pairs

Let Σ be an infinite shift. Then Σ admits an asymptotic pair, and so $CA(\Sigma)$ is non-empty. We construct an $\mathrm{Aut}(\Sigma)$ action on $CA(\Sigma)$. Given an automorphism φ of Σ, define $\hat{\varphi}\colon CA(\Sigma) \to CA(\Sigma)$ by

$$\hat{\varphi}(x, y) = C(\varphi x, \varphi y). \tag{3.1}$$

That is, given a calibrated asymptotic pair (x, y), $\hat{\varphi}$ applies φ to both x and y, and then shifts the resulting asymptotic pair so that it is again calibrated. The next few claims show that this is a well defined action by homeomorphisms. While $\hat{\varphi}$ is easily seen to be measurable, its continuity is less apparent.

By the Curtis-Lyndon-Hedlund Theorem [12], for every $\varphi \in \mathrm{Aut}(\Sigma)$ there is a *memory* $k \in \mathbb{N}$ and a *block map* $B_\varphi\colon A^{\{-k,\dots,k\}} \to A$ such that $[\varphi x]_m = B_\varphi(x_{m-k}, \dots, x_{m+k})$. Importantly, $[\varphi x]_m$ is determined by $(x_{m-k}, \dots, x_{m+k})$. The following claim, which shows that φ is well defined, is a direct consequence.

Claim 3.1. *If (x, y) is an asymptotic pair in Σ then so is $(\varphi x, \varphi y)$, for any $\varphi \in \mathrm{Aut}(\Sigma)$.*

Proof. Let k be a memory of φ. Then $[\varphi x]_m = [\varphi y]_m$ for all $m < M(x, y) - k$, since $(x_{m-k}, \dots, x_{m+k}) = (y_{m-k}, \dots, y_{m+k})$ for such m. And $\varphi x \neq \varphi y$, since φ is a bijection, and since $x \neq y$. $\square$

The next claim offers a bound on the difference between $M(x, y)$ and $M(\varphi x, \varphi y)$. This will be the key component in the proof that $\hat{\varphi}$ is continuous.

Claim 3.2. *For every $\varphi \in \mathrm{Aut}(\Sigma)$ there is a $B \in \mathbb{N}$ such that $|M(x, y) - M(\varphi x, \varphi y)| \leq B$ for every asymptotic pair (x, y) in Σ.*

Proof. Let k be a memory of φ. Then $[\varphi x]_m = [\varphi y]_m$ for all $m < M(x, y) - k$, as noted in the proof of Claim 3.1 above. Hence

$$M(\varphi x, \varphi y) \geq M(x, y) - k,$$

which provides one side of the desired inequality.

Now, let k' be a memory of φ^{-1}. Then by the same argument applied to the pair $(\varphi x, \varphi y)$ and the automorphism φ^{-1} we have that

$$M(\varphi^{-1}\varphi x, \varphi^{-1}\varphi y) \geq M(\varphi x, \varphi y) - k',$$

which provides the other side of the inequality. Thus the claim holds for $B = \max\{k, k'\}$. □

Proposition 3.3. *Each map* $\hat{\varphi}\colon CA(\Sigma) \to CA(\Sigma)$ *is continuous.*

Proof. Fix $\varphi \in \mathrm{Aut}(\Sigma)$. By definition, $M(x, y) = 0$ for $(x, y) \in CA(\Sigma)$. Thus, by Claim 3.2, there is some B such that $|M(\varphi x, \varphi y)| \leq B$ for all $(x, y) \in CA(\Sigma)$. Thus the map $M_\varphi\colon CA(\Sigma) \to \mathbb{Z}$ given by $M_\varphi(x, y) = M(\varphi x, \varphi y)$ takes values in $\{-B, \ldots, B\}$. Furthermore

$$M_\varphi(x, y) = \min\{m \in \{-B, \ldots, B\} : [\varphi x]_m \neq [\varphi y]_m\},$$

and so M_φ is continuous, since φ is continuous. Since

$$\hat{\varphi}(x, y) = (\sigma^{-M_\varphi(x,y)}\varphi x, \sigma^{-M_\varphi(x,y)}\varphi y),$$

and again using that φ is continuous, it follows that $\hat{\varphi}$ is also continuous. □

The proof above in fact shows a stronger claim, which will be important later:

Claim 3.4. *For each* $\varphi \in \mathrm{Aut}(\Sigma)$ *there is a* $b \in \mathbb{N}$ *such that the* m*th coordinates of* $\hat{\varphi}(x, y)$ *depend only on* $(x_{-m-b}, \ldots, x_{m+b})$ *and* $(y_{-m-b}, \ldots, y_{m+b})$.

Finally, the next claim completes the proof that we have defined an action of $\mathrm{Aut}(\Sigma)$ on $CA(\Sigma)$.

Claim 3.5. *The map* $\varphi \mapsto \hat{\varphi}$ *is a group homomorphism from* $\mathrm{Aut}(\Sigma)$ *to* $\mathrm{Homeo}(CA(\Sigma))$.

Proof. By Proposition 3.3, each $\hat{\varphi}$ is a homeomorphism of the compact space $CA(\Sigma)$. It thus suffices to prove that $\hat{\rho}\hat{\varphi} = \widehat{\rho\varphi}$ for all $\rho, \varphi \in \mathrm{Aut}(\Sigma)$.

By definition

$$\hat{\rho}\hat{\varphi}(x, y) = \hat{\rho}(\sigma^m\varphi x, \sigma^m\varphi y)$$

where $m = -M(\varphi x, \varphi y)$. Applying the definition again we get

$$\hat{\rho}\hat{\varphi}(x, y) = (\sigma^n\rho\sigma^m\varphi x, \sigma^n\rho\sigma^m\varphi y),$$

where $n = -M(\rho\sigma^m\varphi x, \rho\sigma^m\varphi y)$. Since ρ commutes with σ, $n = -M(\sigma^m\rho\varphi x, \sigma^m\rho\varphi y)$, and by (2.1), $n = -M(\rho\varphi x, \rho\varphi y) - m$. Thus, and by again using the fact that σ and ρ commute,

$$\begin{aligned}\hat{\rho}\hat{\varphi}(x, y) &= (\sigma^{n+m}\rho\varphi x, \sigma^{n+m}\rho\varphi y)\\ &= (\sigma^{-M(\rho\varphi x,\rho\varphi y)}\rho\varphi x, \sigma^{-M(\rho\varphi x,\rho\varphi y)}\rho\varphi y)\\ &= \widehat{\rho\varphi}(x, y).\end{aligned}$$

□

4. The drift cocycle and drift homomorphisms

Define the *drift cocycle* $c\colon \mathrm{Aut}(\Sigma) \times CA(\Sigma) \to \mathbb{Z}$ by

$$c(\varphi, (x, y)) = M(\varphi x, \varphi y).$$

In a sense, $c(\varphi, (x, y))$ captures the amount by which φ shifts the asymptotic pair (x, y). In particular, by (2.1), $c(\sigma, (x, y)) = 1$ for all $(x, y) \in CA(\Sigma)$.

Claim 4.1. *The drift cocycle c is continuous and satisfies the cocycle relation*

$$c(\varphi\rho, (x, y)) = c(\varphi, \hat{\rho}(x, y)) + c(\hat{\rho}, (x, y)).$$

Proof. In the proof of Proposition 3.3 we defined $M_\varphi(x, y) = M(\varphi x, \varphi y) = c(\varphi, (x, y))$ and showed that it is continuous, and thus c is continuous. It remains to be shown that it satisfies the cocycle relation.

By definition,

$$\begin{aligned} c(\varphi, \rho(x, y)) &= c(\varphi, C(\rho x, \rho y)) \\ &= c(\varphi, (\sigma^{-M(\rho x, \rho y)}\rho x, \sigma^{-M(\rho x, \rho y)}\rho y)) \\ &= M(\varphi\sigma^{-M(\rho x, \rho y)}\rho x, \varphi\sigma^{-M(\rho x, \rho y)}\rho y). \end{aligned}$$

Since φ commutes with σ, we have that

$$c(\varphi, \rho(x, y)) = M(\sigma^{-M(\rho x, \rho y)}\varphi\rho x, \sigma^{-M(\rho x, \rho y)}\varphi\rho y).$$

Applying (3.1) yields

$$c(\varphi, \rho(x, y)) = M(\varphi\rho x, \varphi\rho y) - M(\rho x, \rho y),$$

which is equal to $c(\varphi\rho, (x, y)) - x(\rho, (x, y))$. □

Since $M(x, y) = 0$ for all $(x, y) \in CA(\Sigma)$, by Claim 3.2 c is a bounded cocycle:

Claim 4.2. *For each $\varphi \in \mathrm{Aut}(\Sigma)$ there exists a $B \in \mathbb{N}$ such that $|c(\varphi, (x, y))| \le B$ for all $(x, y) \in CA(\Sigma)$.*

Suppose ν is a Borel probability measure on $CA(\Sigma)$ that is $\mathrm{Aut}(\Sigma)$-invariant, i.e., $\nu(\hat{\varphi}(A)) = \nu(A)$ for every $\varphi \in \mathrm{Aut}(\Sigma)$ and Borel $A \subset CA(\Sigma)$. Let $\Phi_\nu\colon \mathrm{Aut}(\Sigma) \to \mathbb{R}$ be given by

$$\Phi_\nu(\varphi) = \int c(\varphi, (x, y))\, \mathrm{d}\nu(x, y).$$

We call Φ_ν a *drift* homomorphism.

Claim 4.3. *Φ_ν is a well defined homomorphism $\mathrm{Aut}(\Sigma) \to \mathbb{R}$, with $\Phi_\nu(\sigma) = 1$.*

Proof. By Claim 4.1 $c(\varphi, \cdot)$ is continuous and thus measurable. By Claim 4.2 it is bounded. Hence it is integrable, and $\Phi_\nu(\varphi)$ is well defined. To show that it is a homomorphism we first apply the cocycle relation, and then the invariance of ν

$$\begin{aligned}
\Phi_\nu(\rho\varphi) &= \int c(\varphi\rho, (x, y))\, \mathrm{d}\nu(x, y) \\
&= \int c(\varphi, \hat{\rho}(x, y))\, \mathrm{d}\nu(x, y) + \int c(\rho, (x, y))\, \mathrm{d}\nu(x, y) \\
&= \int c(\varphi, (x, y))\, \mathrm{d}\nu(x, y) + \int c(\rho, (x, y))\, \mathrm{d}\nu(x, y) \\
&= \Phi_\nu(\varphi) + \Phi_\nu(\rho).
\end{aligned}$$

Finally, $\Phi_\nu(\sigma) = 1$, since $c(\sigma, (x, y)) = 1$ for all (x, y). □

In light of this claim, we prove our main theorem by showing that if Σ is zero entropy shift with an asymptotic point then $CA(\Sigma)$ admits an $\mathrm{Aut}(\Sigma)$-invariant measure. This is what we do in the next section.

5. $\mathrm{Aut}(\Sigma)$-invariant random calibrated asymptotic pairs

In this section we construct, for each zero entropy shift Σ, a Borel probability measure on $CA(\Sigma)$ that is $\mathrm{Aut}(\Sigma)$-invariant. This construction is nearly identical to that of the main result in [11]; we provide the details for completeness.

Let $W_n \subseteq A^{2n+1} \times A^{2n+1}$ denote the set of *word-pairs* that appear in the centered window of radius n in $CA(\Sigma)$. That is, W_n is the set of $(w_1, w_2) \in A^{2n+1} \times A^{2n+1}$ such that $w_1 = (x_{-n}, \ldots, x_n)$ and $w_2 = (y_{-n}, \ldots, y_n)$ for some $(x, y) \in CA(\Sigma)$. We say that (x, y) *projects* to $(w_1, w_2) \in W_n$ if (w_1, w_2) appears in (x, y), and denote $\pi_n(x, y) = (w_1, w_2)$, $\pi_n \colon CA(\Sigma) \to W_n$. As each $(w_1, w_2) \in W_n$ appears in some pair $(x, y) \in CA(\Sigma)$, we can find a set $\bar{W}_n \subseteq CA(\Sigma)$ of the same size as W_n, and where each $(w_1, w_2) \in W_n$ appears in exactly one $(x, y) \in \bar{W}_n$. I.e., π_n restricted to $\bar{W}_n$ is a bijection.

Since $|W_n| \leq (P_\Sigma(2n+1))^2$, and since Σ has zero entropy, $|W_n|$ grows sub-exponentially, and so there is a sequence $(n_m)_m$ such that

$$\frac{|W_{n_m+m}|}{|W_{n_m}|} \leq 1 + o_m(1).$$

That is, along the sequence n_m, the number of word-pairs in a window of width n_m+m is only a small fraction more than in a window of length n_m. It follows that, along this sequence, at least a $1 - o(1)$ fraction of the word-pairs in W_{n_m} have a unique extension to a word-pair in W_{n_m+m}. Denote such a sequence of sets by U_m.

Let ν_m be the uniform measure on $\bar{W}_{n_m}$, which, we remind the reader, is a subset of $CA(\Sigma)$ for which the projection to the words W_{n_m} is a bijection. Since $CA(\Sigma)$ is compact, the sequence ν_m has a subsequential limit ν.

Proposition 5.1. *The measure ν on $CA(\Sigma)$ is* $\mathrm{Aut}(\Sigma)$*-invariant.*

Proof. Since ν is defined on the Borel sigma-algebra, to show that it is invariant it suffices to show that $\nu(\hat{\varphi}^{-1}\bar{E}) = \nu(\bar{E})$ for every $\varphi \in \mathrm{Aut}(\Sigma)$ and every clopen $\bar{E} \subseteq CA(\Sigma)$.

Since $\bar{E}$ is clopen, for all m large enough there is a set $E_m \subset W_{n_m}$ such that $\bar{E}$ is the set of all $(x, y) \in CA(\Sigma)$ that project to some $(w_1, w_2) \in E_m$. That is, $\bar{E} = \pi_{n_m}^{-1}(E_m)$. Hence $|\bar{E} \cap \bar{W}_{n_m}| = |E_m \cap W_{n_m}| = |E_m|$, where the last equality holds since $E_m \subseteq W_{n_m}$. It follows that

$$\nu_m(\bar{E}) = \frac{|\bar{E} \cap \bar{W}_{n_m}|}{|\bar{W}_{n_m}|} = \frac{|E_m \cap W_{n_m}|}{|W_{n_m}|} = \frac{|E_m|}{|W_{n_m}|},$$

where the first equality is the definition of ν_m. Likewise,

$$\nu_m(\hat{\varphi}^{-1}\bar{E}) = \frac{|(\hat{\varphi}^{-1}\bar{E}) \cap \bar{W}_{n_m}|}{|\bar{W}_{n_m}|} = \frac{|\bar{E} \cap \hat{\varphi}\bar{W}_{n_m}|}{|\bar{W}_{n_m}|}.$$

Now, $\bar{E} \cap \hat{\varphi}\bar{W}_{n_m}$ is the set of elements of $\hat{\varphi}\bar{W}_{n_m}$ that are in $\bar{E}$. Since $\bar{E} = \pi_{n_m}^{-1}(E_m)$, The size of this set is equal to the number of elements $(x, y) \in \hat{\varphi}\bar{W}_{n_m}$ such that $\pi_{n_m}(x, y)$ is in E_m:

$$\nu_m(\hat{\varphi}^{-1}\bar{E}) = \frac{|\{(x, y) \in \hat{\varphi}\bar{W}_{n_m} \,:\, \pi_{n_m}(x, y) \in E_m\}|}{|\bar{W}_{n_m}|}.$$

By Claim 3.4 there is some b such that for all n, coordinates $\{-n, \ldots, n\}$ of both $\hat{\varphi}(x, y)$ and $\hat{\varphi}^{-1}(x, y)$ are determined by coordinates $(-n - b, \ldots, n + b)$ of (x, y). For $m > b$, the set of word-pairs $U_m \subseteq W_{n_m}$ that have a unique extension to W_{n_m+m} is at least a $1 - o(1)$ fraction of the word-pairs in W_{n_m}. Let $\bar{U}_m \subset \bar{W}_{n_m}$ be the subset of $\bar{W}_{n_m}$ that projects to U_m. Since $\bar{U}_m$ is almost all of $\bar{W}_{n_m}$, we can substitute it for $\bar{W}_{n_m}$ and only incur a vanishing error:

$$\nu_m(\hat{\varphi}^{-1}\bar{E}) = \frac{|\{(x, y) \in \hat{\varphi}\bar{U}_m \,:\, \pi_{n_m}(x, y) \in E_m\}|}{|\bar{W}_{n_m}|} + o(1).$$

The key observation is that because of the unique extension property, for $(w_1, w_2) \in U_m$ there is a (w'_1, w'_2) such that (x, y) projects to (w_1, w_2) if and only if $\hat{\varphi}(x, y)$ projects to (w'_1, w'_2). This holds because these projections are determined by $(x_{-n_m-b}, \ldots, x_{n_m+b})$, which by the unique extension property is determined by $(x_{-n_m}, \dots, x_{n_m})$. Hence the restriction of π_{n_m} to $\hat{\varphi}\bar{U}_m$ is injective. Hence

$$\nu_m(\hat{\varphi}^{-1}\bar{E}) = \frac{|E_m \cap \pi_{n_m}(\hat{\varphi}\bar{U}_m)|}{|W_{n_m}|} + o(1).$$

Since $\bar{U}_m$ includes $1 - o(1)$ of the elements of $\bar{W}_{m_n}$, and since $\pi_{n_m} \circ \hat{\varphi}$ is injective on it, we can replace $\bar{U}_m$ by W_{n_m} while again only incurring an additional vanishing error:

$$\nu_m(\hat{\varphi}^{-1}\bar{E}) = \frac{|E_m \cap W_{n_m}|}{|W_{n_m}|} + o(1).$$

Finally, E_m is a subset of W_{n_m}, and so

$$\nu_m(\hat{\varphi}^{-1}\bar{E}) = \frac{|E_m|}{|W_{n_m}|} + o(1) = \nu_m(\bar{E}) + o(1).$$

By taking the limit $m \to \infty$ it follows that $\nu(\hat{\varphi}^{-1}\bar{E}) = \nu(\bar{E})$, and so ν is $\hat{\varphi}$-invariant. □

6. Proof of Theorem 1

The proof of Theorem 1 is an immediate consequence of Claim 3.5 and Proposition 5.1: By Proposition 5.1, there is an $\mathrm{Aut}(\Sigma)$-invariant probability measure on $CA(\Sigma)$, provided that $CA(\Sigma)$ is non-empty, which holds if Σ no periodic points, and provided that Σ is zero entropy. By Claim 3.5, there exists an associated drift homomorphism Φ_ν satisfies $\Phi_\nu(\sigma) = 1$.

7. Examples and further notes

The author would like to thank Joshua Frisch and Ville Salo for drawing his attention to the following examples.

As an example of an asymptotic pair, let $S \subset \{0,1\}^{\mathbb{Z}}$ be the *sunny side up* shift: $S = \{x \in \{0,1\}^{\mathbb{Z}} : \sum_n x_n \leq 1\}$. That is, S is the σ-orbit closure of $\bar{x}$, where $\bar{x}_0 = 1$ and $\bar{x}_n = 0$ for $n \neq 0$. Let $\bar{z}_n = 0$ for all n. Then $(\bar{x}, \bar{z})$ is a calibrated asymptotic pair in S. In fact, there is only one more: $(\bar{z}, \bar{x})$.

The *topological full group* of Σ is the set of homeomorphisms ϕ of Σ for which there exists a continuous *orbit cocycle* $N_\phi \colon \Sigma \to \mathbb{Z}$ for which $\phi(x) = \sigma^{N_\phi(x)}(x)$ (see, e.g., Katzlinger's survey [14]). Topological full groups admit drift homomorphisms that closely resemble our construction: they arise as expectations of the orbit cocycle N_ϕ with respect to a shift-invariant measure on Σ.

Indeed, the relation to this paper can be made more explicit. Given any shift Σ, let $\Sigma' = \Sigma \times S$. The topological full group embeds as a subgroup of $\mathrm{Aut}(\Sigma')$: given an element of the full group ϕ, define $\bar{\phi} \in \mathrm{Aut}(\Sigma')$ as follows. An element of Σ' is either of the form $(y, \sigma^m \bar{x})$ for $m \in \mathbb{Z}$ or of the form $(y, \bar{z})$. In the latter case let $\bar{\phi}(y, \bar{z}) = (y, \bar{z})$. In the former case let

$$\bar{\phi}(y, \sigma^m \bar{x}) = (y, \sigma^{m+N_\phi(\sigma^{-m} y)} \bar{x}).$$

Consider calibrated asymptotic pairs of the form $((y, \bar{x}), (y, \bar{z})) \in CA(\Sigma')$. On such pairs, the drift cocycle c equals to N_ϕ. Hence, if we choose y at random according to a shift-invariant probability measure on Σ, the expectation of the drift cocycle $c(\phi, ((y, \bar{x}), (y, \bar{z})))$ will equal the expectation of the orbit cocycle $N_\phi(y)$.

Yet another similar construction of a drift homomorphism (which we will not explain in detail) is the "average movement" of Turing machines of shifts, as defined by Barbieri, Kari and Salo [2]; this is a generalization of the drift homomorphisms of full groups.

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