ROBUST MARKET INTERVENTIONS

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ABSTRACT. When can interventions in markets be designed to increase surplus *robustly*—i.e., with high probability—accounting for uncertainty due to imprecise information about economic primitives? In a setting with many strategic firms, each possessing some market power, we present conditions for such interventions to exist. The key condition, *recoverable structure*, requires large-scale complementarities among families of products. The analysis works by decomposing the incidence of interventions in terms of principal components of a Slutsky matrix. Under recoverable structure, a noisy signal of this matrix reveals enough about these principal components to design robust interventions. Our results demonstrate the usefulness of spectral methods for analyzing imperfectly observed strategic interactions with many agents.

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1. Introduction

Market power has recently attracted renewed attention and is thought to have significant and growing welfare implications (see, e.g., Syverson, 2019). While many applied analyses of competition confine the analysis of market power to tightly defined product markets, it is becoming clear that important welfare-relevant spillovers operate *across* such markets (Baqaee and Farhi, 2020; Azar and Vives, 2021; Pellegrino, 2021; Ederer and Pellegrino, 2021). Our theoretical understanding of such spillovers in environments with many firms interacting via general demand systems remains limited. This paper is about the welfare theory of such interactions.

Consider many profit-maximizing, single-product firms simultaneously setting prices, with arbitrary complementarities and substitutabilities across products. For instance, one firm's product—e.g., a Samsung smartphone—may be a substitute to some products—e.g., Apple smartphones—and a complement to others—compatible accessories such as earbuds, watches, and smart home appliances, which may in turn be substitutes or complements to one another.

We are interested in the nature of inefficiencies in such an environment and policies to respond to them. We consider these issues from the perspective of an authority that recognizes the possibility of inefficiency due to market power and can intervene through taxes and subsidies on firms' sales. For concreteness, we will think of this authority as the operator of a large marketplace, such as Amazon, that mediates retail sales and aims to increase the equilibrium economic surplus generated by the marketplace.[1] What principles should guide the design of such interventions? When can such policies be implemented under realistic uncertainty about market parameters?

What makes the problem challenging is that, once we broaden our perspective beyond one traditionally defined market (e.g., smartphones) and consider spillovers to a variety of other complements and substitutes, there is a kind of curse of dimensionality. In a marketplace with numerous and changing goods, the demand system is high-dimensional and a priori unstructured. As any firm's cost changes, the number of potential effects to consider is equal to the number of products; thus, the number of interactions quadratically in this number. Realistic signals will leave substantial uncertainty about many aspects of the structure of the game among the firms (see Section 7.1 for a detailed discussion). In particular, the authority will have nothing close to a precise estimate of the entire demand system. This raises the question is whether there are policies that robustly improve surplus despite this uncertainty.

Our main result is that if demand satisfies a property that we call *recoverable structure*, then there are feasible intervention rules that robustly (i.e., with high probability) increase equilibrium total surplus despite large errors in observing every detail of the system. Moreover, within a natural class of interventions—those that do not reduce consumer surplus[2]—our feasible interventions achieve

[1]The aim of this might be to deliver more value to consumers and producers to keep the platform competitive, or to reclaim the surplus via lump-sum fixed fees.

[2]This constraint is natural for a platform that could lose customers to other marketplaces or a government that can lose political support.

the largest gain in surplus that is possible for a given level of subsidy expenditure. Hence, within this class, these interventions are as good as those that could be designed by an authority with perfect information.

Furthermore, our results provide tight conditions for robust intervention in the following sense. First, the property of recoverable structure cannot be dispensed with; there are reasonable demand systems without recoverable structure for which the authority cannot robustly increase total surplus. Second, we show that there are settings with recoverable structure in which *any* intervention rule that robustly increases total surplus is equivalent, in terms of how it allocates surplus, to the interventions identified by our main result. These tightness results show that robust interventions must, in general, be tailored to the marketplace—there is no simple rule of thumb that always works. On the other hand, under conditions that we identify, it is possible to tailor rules well, despite large uncertainty in many aspects of demand.

The key condition in the paper is recoverable structure. We now explain what it means for demand to have recoverable structure and how this property is used in the construction of robust interventions.

The demand structure is encoded in a matrix $\boldsymbol{D}$ of demand derivatives, which in our setting is equal to the Slutsky matrix. A given cell D_{ij} in this matrix is the derivative of product i's demand with respect to product j's price. Thus, the matrix specifies the complementarity and substitutability relationships across products. Mathematically, the recoverable structure property requires that $\boldsymbol{D}$ can be written as a rank-one matrix with large norm plus a matrix orthogonal to this. This rank-one piece can be thought of as a large principal component: a part of the demand system described by a single vector that accounts for a large amount of demand behavior. In terms of the economic intuition, we will show that recoverable structure entails substantial large-scale complementarities. This manifests as the ability of small subsidies to have large spillover effects that raise the consumption of many goods by significant amounts.

The key point here is that recoverable structure is large-scale structure. When large-scale complementarities are present, they correspond to marketplace-wide double marginalization problems, where many goods all exert externalities on one another. Such broad externalities create the potential for interventions that substantially improve welfare.

The statistical implications of recoverable structure are then central to actually taking advantage of this potential when the Slutsky matrix is observed imperfectly. The authority's signal consists of noisy estimates of the entries of this matrix, with noise magnitudes in each entry comparable to the entries themselves. This noise creates large uncertainty in the operation of a given intervention. We show that, nevertheless, in large markets with recoverable structure, such noisy observation of $\boldsymbol{D}$ can be used to precisely predict the effects of *some* well-chosen interventions—specifically, those operating in the space of eigenvectors associated with the largest eigenvalues of $\boldsymbol{D}$. The key tool for this is the Davis–Kahan theorem (Davis and Kahan, 1970).

Combining the economic and statistical implications of recoverable structure allows us to establish our main result. In markets possessing such structure, the authority can recover precise information about large-eigenvalue components of the spectral decomposition, and interventions based on this information have highly predictable surplus implications.

At a technical level, to perform this analysis we develop a new spectral description of the pass-through of an intervention. That is, we diagonalize the Slutsky matrix to obtain a specific orthonormal basis in which we can express the implications of any intervention as a linear combination of orthogonal effects. These effects correspond to the projection of the intervention onto each eigenvector of the Slutsky matrix $\boldsymbol{D}$. By characterizing the pass-throughs of subsidies to prices, quantities, and welfare separately across these principal components, we are able to prove that targeting the high-eigenvalue principal components yields precisely predictable results achieving our claimed welfare properties. The spectral decomposition may be of independent interest, yielding a useful basis in which price and welfare pass-throughs of cost shocks behave intuitively despite the complexity of a system with arbitrary spillovers.

1.1. **Related literature.** Our paper contributes to the literature on the structure and theoretical properties of market power. For an early theoretical paper, see Dixit (1986); more recent studies include, for example, Vives (1999), Azar and Vives (2021), Nocke and Schutz (2018), and Nocke and Whinston (2022). A recent strand of research in macroeconomics and industrial organization uses differentiated oligopoly network models—similar to the one we consider here—to provide empirical estimates of efficiency losses due to market power (e.g., Pellegrino (2021) and Ederer and Pellegrino (2021)).[3]

Given these estimates of inefficiencies, a natural theoretical question is: What feasible interventions can improve welfare? Our main contribution is to analyze interventions from the perspective of an authority uncertain about the demand structure. Our analysis combines new spectral pass-through formulas with results building on the statistical theory of large matrices, and we identify conditions on the demand structure that ensure the robust achievement of welfare improvements even when many aspects of the demand structure cannot be accurately estimated.[4] This approach has significant implications for understanding which kinds of empirical models are needed to design interventions in large markets with many goods. We elaborate on these issues in Section 7.2.

Methods in high-dimensional statistics are currently attracting considerable interest in econometric settings with high-dimensional covariates (see, e.g., Athey, Bayati, Doudchenko, Imbens, and Khosravi (2021) and Chernozhukov, Hansen, Liao, and Zhu (2023)), and there is work applying related statistical models to informational or behavioral spillovers in social networks (Golub and Jackson, 2012; Dasaratha, 2020; Cai, 2022; Parise and Ozdaglar, 2023; Chandrasekhar, Goldsmith-Pinkham, McCormick, Thau, and Wei, 2024). However, we know little about when noisy data can be effectively used in order to implement desirable interventions in the presence of strategic spillovers, particularly in market settings. We show that, in a large oligopoly market, the concepts developed in the literature on large network recovery can be useful for designing socially desirable interventions.

[3] See also Elliott and Galeotti (2019) for related arguments about how network methods can be useful for competition authorities in developing antitrust investigations.

[4] Our focus on pass-through builds on work emphasizing the value of pass-through as a conceptual tool, e.g., Marshall (1890), Pigou (1920), Dixit (1979) and, more recently, Weyl and Fabinger (2013), Miklos-Thal and Shaffer (2021) and Norris (2024).

Our paper contributes to the theory of network interventions. Early contributions include Borgatti (2006), Ballester, Calvó-Armengol, and Zenou (2006), and Goyal (1996).[5] Spectral methods have recently been applied to optimal intervention problems when spillovers are known (Galeotti, Golub, and Goyal, 2020; Gaitonde, Kleinberg, and Tardos, 2021; Liu and Tsyvinski, 2024).[6] By contrast, in the present paper, the authority observes strategic spillovers with significant noise.[7] The methods we develop for robust interventions can be applied to other network games more generally and we briefly discuss this in Section 7.3. Our analysis of perturbations of taxes and subsidies is related to the classic "tax reform approach" in public finance (Feldstein, 1976; Tirole and Guesnerie, 1981); the study of uncertain spillovers distinguishes our work.

Our approach to robustness is conceptually related to, but methodologically distinct from, an extensive literature in economic theory. That literature focuses on understanding the design of mechanisms and contracts that achieve desired outcomes even when assumptions about the environment (e.g., agents' preferences, beliefs, and rationality) are relaxed; see Carroll (2019) for a survey. Our definition of robustness aligns with the spirit of this literature. However, in our context, the motivation for analyzing robust interventions arises from the high-dimensional nature of the market state, and use methods that align with statistical work in this type of setting.

2. FRAMEWORK

In this section, we present the framework for our study. The foundation is a simple differentiated oligopoly game. Within this game, we introduce a class of interventions available to the authority and calculate the surplus outcomes associated with these interventions. Finally, we introduce the statistical framework describing the signals available to the authority and a notion of rules that use these signals to achieve good outcomes robustly.

2.1. **The differentiated oligopoly game.**

2.1.1. *Demand side.* There is a set $\{1, \dots, n\}$ of distinct products. The demand for these products arises from the consumption choices of a fixed, finite number of optimizing households. Each household $h \in \{1, \dots, H\}$ takes prices as given

[5]The literature on this subject is very large. Intervention design has been studied in models of information diffusion, advertising, finance, security, and pricing, among other topics—see e.g., Banerjee, Chandrasekhar, Duflo, and Jackson (2013), Bloch and Querou (2013), Candogan, Bimpikis, and Ozdaglar (2012), Belhaj and Deroian (2017), Demange (2017), Dziubinski and Goyal (2017), Galeotti and Goyal (2009), and Leduc, Jackson, and Johari (2017).

[6]Some recent work uses spectral analysis to derive conditions for core-selecting reallocative auctions (Rostek and Yoder, 2023), and robust implementation (Ollár and Penta, 2023). See also Aguiar and Serrano (2017) on spectral methods to study Slutsky matrices in a consumer theory setting.

[7]We share with several prior papers the idea that decision-makers act under partial information about the network. For instance, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) study large network games where players have incomplete information about the network structure, described by a random graph; Akbarpour, Malladi, and Saberi (2020) considers seeding in a large random graph; the diffusion process there lacks any form of complementarity. Our questions and methods of analysis are very different from these papers.

and has a *choice utility* that is quasilinear in a numeraire m,

$$U^h(\tilde{\boldsymbol{q}}^h, m) = V^h(\tilde{\boldsymbol{q}}^h) + m,$$

where V^h is a twice-differentiable and strictly concave function of the consumption profile $\tilde{\boldsymbol{q}}^h \in \mathbb{R}^n$ and m is a numeraire ("money"), in which all prices are denominated. Given a price profile $\tilde{\boldsymbol{p}}$, the household's problem is to choose a bundle $\tilde{\boldsymbol{q}}^h$ to maximize $U^h(\tilde{\boldsymbol{q}}^h, m) - \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{q}}^h$. Letting $\boldsymbol{q}^h(\tilde{\boldsymbol{p}})$ be the solution to household h's problem (unique by concavity of V^h), total market demand is[8]

$$\boldsymbol{q}(\tilde{\boldsymbol{p}}) = \sum_{h=1}^{H} \boldsymbol{q}^h(\tilde{\boldsymbol{p}}).$$

Note that we use tilde notation for an arbitrary price or quantity, and then drop the tilde for these variables to indicate some optimal or equilibrium solution.

2.1.2. *Supply side.* There is a firm associated with each product: Firm i produces product i. Firms play a simultaneous pricing game; each firm chooses $\tilde{p}_i \geq 0$. For any realized profile of prices $\tilde{\boldsymbol{p}}$, firm i's profit is

$$q_i(\tilde{\boldsymbol{p}})(\tilde{p}_i - c_i), \tag{1}$$

where c_i is the (constant) marginal cost of production.

We fix a vector $\boldsymbol{c}^0$ of marginal costs and a pure-strategy Nash equilibrium $\boldsymbol{p}^0$, and we refer to these as the *status quo* marginal costs and equilibrium, respectively.[9] To facilitate unambiguous local comparative statics, we make the following assumption.

Assumption 1 (Local equilibrium uniqueness)**.** There exist $\nu > 0$ and $\rho > 0$ such that, for all $\boldsymbol{c}$ with $\|\boldsymbol{c} - \boldsymbol{c}^0\| < \nu$, there is a unique pure-strategy Nash equilibrium $\boldsymbol{p}(\boldsymbol{c})$ in a ρ-neighborhood of $\boldsymbol{p}^0$.

From now on, we confine attention to cost perturbations within the set discussed in Assumption 1, and when we refer to an *equilibrium* at any cost profile, we mean the locally unique one entailed by this assumption.

2.2. **Interventions and their effects.** An *authority*—an institution that oversees a marketplace—can intervene in the market. We focus on a canonical set of interventions: per-unit subsidies and taxes. For a consumption profile $\tilde{\boldsymbol{q}}$, a *per-unit subsidy intervention*

$$\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$$

consists of a transfer $\sigma_i \tilde{q}_i$ from the authority to firm i; a positive σ_i corresponds to a subsidy to firm i, while a negative σ_i corresponds to a tax.

[8]Because the households' utilities are quasilinear in money, one can derive the same aggregate demand from a representative consumer, and this description is sufficient for studying producer and aggregate consumer surplus. However, some of our results will provide more refined results about the effects on individual consumers, showing that no household is significantly hurt.

[9]Existence of a pure strategy equilibrium is guaranteed if profits are quasi-concave in prices (see Theorem 1.2 in Fudenberg and Tirole (1991)). A sufficient condition for this is that the functions $1/q_i(\tilde{\boldsymbol{p}})$ are convex in $\tilde{p}_i$ (Vives (1999) page 149). This holds, for instance, under the much stronger condition that demand is linear in prices; in this case the equilibrium is also unique. The general sufficient condition for local uniqueness is nonsingularity of the Jacobian of best responses at equilibrium, which will hold generically in our setting (McLennan, 2018).

We will focus on demand functions that are linear in a neighborhood of the status quo equilibrium, as captured by the following assumption.

Assumption 2 (Linearity of demand locally)**.** There exists $\rho > 0$ such that, in a neighborhood of radius ρ around the initial equilibrium $\boldsymbol{p}^0$, the demand function $q_i(\tilde{\boldsymbol{p}})$ is linear in $\tilde{p}_i$ for every firm i.

This assumption facilitates our analysis of interventions, yielding simple formulas for comparative statics. It also captures much of the economics of our robust interventions problem, despite its simplicity. We discuss how to extend the analysis for the case of non-linear demand in Section 7.4.

The firms' first-order conditions imply that equilibrium prices $\boldsymbol{p}$ around the status quo $\boldsymbol{p}^0$ satisfy

$$q_i(\boldsymbol{p}) = -\frac{\partial q_i}{\partial p_i}(\boldsymbol{p})(p_i - c_i).$$

The linearity assumption implies that $\frac{\partial q_i}{\partial p_i}(\boldsymbol{p})$ remains constant when prices change locally around $\boldsymbol{p}^0$. Hence, we can replace the $\boldsymbol{p}$-dependent partial derivative in the above equation with the constant $\frac{\partial q_i}{\partial p_i}(\boldsymbol{p}^0)$. By strict concavity of the consumer utility functions, this is a negative number; from now on, we maintain a normalization (by choosing suitable units in which to express the quantity produced by firm i) that $\frac{\partial q_i}{\partial p_i}(\boldsymbol{p}^0) = -1$ (see Appendix A.1). After this normalization, equilibrium behavior is summarized by the following system of equations:

$$\boldsymbol{q}(\boldsymbol{p}) = \boldsymbol{p} - \boldsymbol{c}. \tag{2}$$

Implicitly differentiating this (linear) system, we obtain that the effect of a small intervention $\boldsymbol{\sigma}$ on prices is determined by the following system of equations:

$$[\boldsymbol{I} - \boldsymbol{D}]\dot{\boldsymbol{p}} = -\boldsymbol{\sigma}, \tag{3}$$

where $\boldsymbol{\sigma} = \boldsymbol{c}^0 - \boldsymbol{c}$ is the intervention (i.e., tax or subsidy offered by the authority), $\dot{\boldsymbol{p}}$ is the derivative of $\boldsymbol{p}$ in the direction of $\boldsymbol{\sigma}$ (see eq. (6)), and $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{p}^0)$ is the Slutsky matrix (in the normalized units):

$$D_{ij} = \frac{\partial q_i}{\partial p_j}(\boldsymbol{p}^0).$$

For $i \neq j$, if $D_{ij} > 0$ (resp. $D_{ij} < 0$) then, around the equilibrium, products i and j are substitutes (resp. complements).[10] Note also that quantity changes following the intervention $\boldsymbol{\sigma}$ are pinned down by

$$\dot{\boldsymbol{q}} = \boldsymbol{D}\dot{\boldsymbol{p}}. \tag{4}$$

The local linearity of demand implies that comparative statics of prices and quantities are fully determined by the Slutsky matrix $\boldsymbol{D}$. We note that $\boldsymbol{D}$ satisfies the following property (Nocke and Schutz, 2017).

Property NSD. The normalized Slutsky matrix $\boldsymbol{D}$ is negative semidefinite, symmetric, and has diagonal entries $D_{ii} = -1$.

[10] In general, the matrix of derivatives of Marshallian demand need not be the same as the Slutsky matrix (which works with compensated demand). However, in this demand system, the wealth effect is zero due to the fact that the goods utility and money are additively separable. Thus, the two matrices coincide (Nocke and Schutz, 2017), and so we use the term "Slutsky matrix" throughout.

This property holds because the demand function can be taken to arise from a representative household (with a twice-differentiable utility function for goods equal to the sum of the consumers' utilities, V^h).

Finally, the following helpful normalization is without loss of generality—it holds by suitably adjusting the units of the numeraire m.

Assumption 3. The quantity vector's Euclidean norm $\|\boldsymbol{q}^0\|$ is at most 1.

In what follows, we maintain Assumptions 1–3 unless stated otherwise.

2.3. **Surpluses.** The authority's net spending associated with an intervention $\boldsymbol{\sigma}$ is denoted by $S = \boldsymbol{\sigma} \cdot \tilde{\boldsymbol{q}}$. Recalling Assumption 1, we define the set of feasible interventions as[11]

$$\boldsymbol{\Sigma} = \{\boldsymbol{\sigma} \in \mathbb{R}^n : \|\boldsymbol{\sigma}\| < \nu\}.$$

The authority cares about the surplus that different market participants obtain in equilibrium. We focus on three canonical metrics: consumer surplus C, producer surplus P, and total surplus W (accounting for the intervention expenditure S). Given an intervention $\boldsymbol{\sigma}$, and quantity profiles $\{\boldsymbol{q}^h\}_{h=1,\ldots,n}$ and $\boldsymbol{q} := \sum_h \boldsymbol{q}^h$, these are:

$$C = \sum_h C^h \quad \text{where} \quad C^h = V^h(\boldsymbol{q}^h) - \boldsymbol{q}^h \cdot \boldsymbol{p},$$

$$P = (\boldsymbol{p} - \boldsymbol{c}) \cdot \boldsymbol{q} \quad \text{and} \quad W = C + P - S. \tag{5}$$

We evaluate the effect of an intervention $\boldsymbol{\sigma}$ on an outcome variable Y by its first derivative. Formally, the first-order effect on any outcome variable Y (e.g., $Y = C$ or P) of changing subsidies in the direction $\boldsymbol{\sigma}$ is defined by[12]

$$\dot{Y}_{\boldsymbol{\sigma}} = \sum_i \frac{dY}{d\sigma_i}\sigma_i. \tag{6}$$

Fixing the parameters of the economy, the set of *possible* surplus outcomes is defined to be the set of tuples

$$\{(\dot{C}_{\boldsymbol{\sigma}}, \dot{P}_{\boldsymbol{\sigma}}, \dot{S}_{\boldsymbol{\sigma}}) : \boldsymbol{\sigma} \in \mathbb{R}^n\}$$

of surplus outcomes corresponding to some intervention. The following proposition characterizes the possible surplus outcomes.

Proposition 1. For generic $\boldsymbol{q}^0$ and any $\boldsymbol{D}$, the following hold:

(1) A surplus outcome $(\dot{C}, \dot{P}, \dot{S})$ is possible if and only if it satisfies

$$\dot{C} + \frac{1}{2}\dot{P} = \dot{S}. \tag{7}$$

(2) If $(\dot{C}, \dot{P}, \dot{S})$ is possible and $\dot{C} \geq 0$, then

$$\dot{C} + \dot{P} \leq 2\dot{S}.$$

[11] We restrict attention to deterministic interventions but this is immaterial to our results.

[12] We often omit the subscript $\boldsymbol{\sigma}$ when there is no ambiguity about the relevant intervention.

Proof sketch. Consumer surplus satisfies the Marshallian formula $\dot{C} = -\boldsymbol{q}^0 \cdot \dot{\boldsymbol{p}}$. Meanwhile, $\dot{P} = \boldsymbol{q}^0 \cdot (\dot{\boldsymbol{p}} + \boldsymbol{\sigma}) + (\boldsymbol{p} - \boldsymbol{c}) \cdot \dot{\boldsymbol{q}} = 2\boldsymbol{q}^0 \cdot \dot{\boldsymbol{q}}$, where we have used the equilibrium condition $\boldsymbol{p} - \boldsymbol{c} = \boldsymbol{q}$ from (2). Finally, $\dot{S} = -\boldsymbol{q}^0 \cdot \dot{\boldsymbol{c}} = \boldsymbol{q}^0 \cdot \dot{\boldsymbol{\sigma}}$ by definition. This establishes that any $(\dot{C}_{\boldsymbol{\sigma}}, \dot{P}_{\boldsymbol{\sigma}}, \dot{S}_{\boldsymbol{\sigma}})$ satisfies (2.3). The proof that *every* tuple satisfying () arises from some $\boldsymbol{\sigma}$ is more involved and is found in Appendix B. The second part follows from the first. □

Part (1) says that, for a given level of spending $\dot{S}$, the set of possible outcomes is a line in $(\dot{P}, \dot{C})$ space. This "Pareto frontier" tells us what is possible in principle, and in particular implies that market surplus cannot be increased by more than twice the level of expenditure without reducing consumer surplus.

While this result characterizes what post-intervention outcomes are possible in equilibrium, it does not discuss how to attain them. The following lemma is an important input in the answer to this question, and is used in the proof of the "if" direction of Proposition 1(1).

Lemma 1. For any intervention $\boldsymbol{\sigma}$, spending is given by $\dot{S}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \cdot \boldsymbol{q}^0$, while

$$\dot{W}_{\boldsymbol{\sigma}} = \frac{1}{2}\dot{P}_{\boldsymbol{\sigma}} = (\boldsymbol{q}^0)^{\mathsf{T}} \boldsymbol{D}[\boldsymbol{I} - \boldsymbol{D}]^{-1}\boldsymbol{\sigma} \tag{8}$$

The proof works by combining the formulas in the proof of the above proposition with the formulas in (3) and (4) to make price and quantity derivatives explicit.

Formula (8) can be interpreted as a pass-through equation: entry i of the row vector $\boldsymbol{w}^{\mathsf{T}} = (\boldsymbol{q}^0)^{\mathsf{T}} \boldsymbol{D}[\boldsymbol{I} - \boldsymbol{D}]^{-1}$ gives the impact on total surplus of increasing the subsidy σ_i. The authority aims to achieve a desired total surplus effect $\dot{W}_{\boldsymbol{\sigma}}$, possibly subject to additional requirements, such as holding spending $\dot{S}_{\boldsymbol{\sigma}}$ constant.

2.4. **The challenge.** Matrix inverses such as $[\boldsymbol{I} - \boldsymbol{D}]^{-1}$ can be extremely sensitive to entries of $\boldsymbol{D}$. Therefore, without precise knowledge of $\boldsymbol{D}$ and $\boldsymbol{q}^0$, the authority may not be able to implement a desired point on the line defined by (2.3). For example, the authority may not be sure that a given intervention will increase total surplus ($\dot{W}_{\boldsymbol{\sigma}} > 0$) rather than decrease it ($\dot{W}_{\boldsymbol{\sigma}} < 0$). Indeed, it seems hard to justify the detailed study of comparative statics such as eq. (8) when n is large without confronting the uncertainty about the ingredients of the formula an analyst or authority is likely to face.

These observations motivate the central question of this paper: *Which interventions have surplus effects that can be predicted with confidence by an authority facing substantial uncertainty about market primitives?*

2.5. **The statistical framework.** To formalize this question, we introduce a simple model of noisy observation of $\boldsymbol{D}$ and $\boldsymbol{q}^0$.

In the linear oligopoly environment, Proposition 1 establishes that to determine the first-order surplus effects of any intervention $\boldsymbol{\sigma}$, the only additional data needed is the tuple $(\boldsymbol{D}, \boldsymbol{q}^0)$, where $\boldsymbol{D}$ is a negative semidefinite matrix with diagonal entries -1 and $\boldsymbol{q}^0$ is a vector of norm at most 1. We call such a tuple a *market state* and denote it by $\boldsymbol{\theta}$. The set of possible market states is denoted by $\boldsymbol{\Theta}$.

The authority receives a signal, denoted by $\widehat{\boldsymbol{\theta}} \in \widehat{\boldsymbol{\Theta}}$, about the market state.[13] This signal consists of random variables

$$\widehat{\boldsymbol{D}} = \boldsymbol{D} + \boldsymbol{E} \text{ and } \widehat{\boldsymbol{q}}^0 = \boldsymbol{q}^0 + \boldsymbol{\varepsilon}.$$

We will later detail assumptions on the error terms. For now, a canonical setting to keep in mind is one where all error draws are mean-zero and independent, with each error having a magnitude comparable to the underlying entry D_{ij} or q_i^0. Note that the signal need not lie in the same set as the state; for example, under our assumptions, $\boldsymbol{D}$ is negative semidefinite, but the signal $\widehat{\boldsymbol{D}}$ might not be.

Let $\varphi_{\boldsymbol{\theta}} \in \Delta(\widehat{\boldsymbol{\Theta}})$ denote the probability measure over signals when the state is $\boldsymbol{\theta}$.

2.6. **Robust intervention rules.** The authority designs an *intervention rule*[14]

$$\boldsymbol{R} : \mathcal{T} \to \boldsymbol{\Sigma},$$

prescribing an intervention $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ for every possible signal $\widehat{\boldsymbol{\theta}}$. We now define a notion of such a rule robustly achieving a desired property.

A *market outcome* is a tuple $(\boldsymbol{\theta}, \boldsymbol{\sigma})$ consisting of a market state and an intervention. We call this pair the outcome because it determines production, consumption, and transfers. A *property* is a measurable subset $\mathscr{P} \subseteq \boldsymbol{\Theta} \times \boldsymbol{\Sigma}$ of all possible outcomes. An important example of a property is increasing total surplus, given by (8):

$$\begin{aligned}\mathscr{P} &= \{(\boldsymbol{\theta}, \boldsymbol{\sigma}) : \dot{W}_{\boldsymbol{\sigma}} > 0\} \\ &= \{((\boldsymbol{D}, \boldsymbol{q}^0), \boldsymbol{\sigma}) : (\boldsymbol{q}^0)^\mathsf{T} \boldsymbol{D}[\boldsymbol{I} - \boldsymbol{D}]^{-1} \boldsymbol{\sigma} > 0\}\end{aligned}$$

We are interested in understanding which properties can be achieved with high probability in all market states that the authority considers possible:

Definition 1. An intervention rule $\boldsymbol{R}$ achieves a property $\mathscr{P}$ ϵ–robustly if the following holds: For every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, we have

$$\varphi_{\boldsymbol{\theta}} \left(\{\widehat{\boldsymbol{\theta}} : (\boldsymbol{\theta}, \boldsymbol{R}(\widehat{\boldsymbol{\theta}})) \in \mathscr{P})\} \right) \geq 1 - \epsilon.$$

The only randomness in the definition is in the signal draw—recall $\varphi_{\boldsymbol{\theta}}$ is the distribution of the signal given the true state $\boldsymbol{\theta}$.

We close with some remarks on our modeling choices.

Remark 1. The main parameter in the definition of robustness is the set $\boldsymbol{\Theta}$ of *possible* market states. If the authority has any prior μ over $\boldsymbol{\Theta}$, then, given conditional error distributions $(\varphi_{\boldsymbol{\theta}})_{\boldsymbol{\theta}}$ and a property $\mathscr{P}$ of interest, the probability $\mathbb{P}_\mu(\mathscr{P})$ can be computed under that prior. Requiring an intervention rule to achieve a property ϵ-robustly is equivalent to requiring $\mathbb{P}_\mu(\mathscr{P}) \geq \epsilon$ for every possible prior μ.

A reason to focus on robust intervention rules is that for primitives such as Slutsky matrices involving a large number of goods, it is not at all obvious how an authority should specify a prior. A set $\boldsymbol{\Theta}$ can, in contrast, encode an assumption on the environment (e.g., $\boldsymbol{D}$ being negative semidefinite) without having to take a stand on prior probabilities.

[13] Section 7.1 discusses some practical examples.

[14] This should be measurable in a suitable sense, which is clear in our application.

Remark 2. Notice that the definition of robust interventions does not depend on any details of the linear oligopoly environment. The definition can be applied in any environment where $\boldsymbol{\theta}$ is some state, $\boldsymbol{\sigma}$ is the authority's choice, the pair $(\boldsymbol{\theta}, \boldsymbol{\sigma})$ fully determines all outcomes of interest to the authority, and there is a known distribution over signals given states.

Nevertheless, the linear oligopoly game and the focus on first-order effects provide a useful starting point: a canonical environment in which rich spillovers can be fully specified by a familiar finite-dimensional object ($\boldsymbol{D}$). Moreover, formulas using only simple matrix operations suffice to describe surplus effects, and these facilitate tractability of statistical questions. Extensions to nonlinear environments are discussed in Section 7.4.

3. Recoverable structure and its economic implications

The main result of this paper says that *under conditions on the set of possible market states* $\boldsymbol{\Theta}$ *and conditions on the distribution of errors, there are intervention rules that improve surplus robustly.* The content of the result lies in specifying the assumptions on $\boldsymbol{\Theta}$ and error distributions. While these conditions are somewhat involved to state in full generality, a useful preview can be presented in an important class of examples, related to the classic stochastic block model in network theory.

3.1. A stochastic block model of demand.

There is a fixed, finite set of product *types*, $M = \{1, 2, \ldots, m\}$. The interactions of products are determined by their types, and given by entries of an m-by-m type-level matrix $\bar{\boldsymbol{D}}$ satisfying Property NSD. Quantities are also determined by types, according to a vector $\bar{\boldsymbol{q}} \in \mathbb{R}^m$. Let the type of good i be $k(i) \in M$.[15] Finally, fixing $\gamma(n) \in (0, 1]$, let

$$D_{ij} = \begin{cases} \gamma(n)\bar{D}_{k(i)k(j)} & i \neq j \\ -1 & i = j \end{cases} \quad \text{and} \quad q_i^0 = \bar{q}_{k(i)}. \tag{9}$$

An explicit example of a stochastic block model appears in Section 5. The errors ε_i in observing $\boldsymbol{q}^0$ and the errors E_{ij} in observing $\boldsymbol{D}$ are i.i.d. and mean zero, with bounded support, satisfying $\mathrm{Var}[E_{ij}] < \overline{\sigma}$ and $\mathrm{Var}[\|\boldsymbol{\varepsilon}\|] < \overline{\varsigma}\|\boldsymbol{q}^0\|$ for some fixed real numbers $\overline{\sigma}$ and $\overline{\varsigma}$. This entails that errors in demand signals and quantity signals are of the same order of magnitude as the underlying parameters. Within this model, we have:

Proposition 2. Fix a generic $(\bar{\boldsymbol{D}}, \bar{\boldsymbol{q}})$ and assume $\gamma(n)n^{1/2} \to \infty$. For large enough n and any spending level s, there is an intervention rule that robustly achieves a surplus outcome arbitrarily close to $(\dot{P}, \dot{C}, \dot{S}) = (2s, 0, s)$.

Note that Proposition 1(2) implies that the intervention rules guaranteed by Proposition 2 achieve the largest possible increase in total surplus subject to not reducing consumer surplus. It is also worth noting that the partitioning of goods into types need not be known in advance to achieve the target surplus outcome.

In this setting, the assumption that $\gamma(n)n^{1/2}$ is large ensures that some of the entries of $\bar{\boldsymbol{D}}$ can be recovered despite noisy observation. Notice, however, that

[15] We let $N(t)$ be the number of products of type t, which depends on the total number of products n, and assume $N(t)/n$ is a convergent sequence as $n \to \infty$ for each $t \in M$.

other entries of $\bar{\boldsymbol{D}}$ may *not* be recoverable precisely. One extreme example of this occurs if some types have only a single good, or more generally a number of goods uniformly bounded in n. The substance of the result is that the information that *can* be recovered about $\bar{\boldsymbol{D}}$ suffices to robustly achieve the indicated outcome.

The result raises several questions: First, how is recoverable information used to design effective interventions, and what determines the limits of this strategy? More fundamentally, how can it be extended beyond the specific structure of the stochastic block model? The assumption that there are arbitrarily large[16] "blocks" of products whose exact relationships to other goods (the entries D_{ij}) are identical within type is restrictive. Are there more flexible structures that permit robust interventions—e.g., in cases where no entry of $\boldsymbol{D}$ can be recovered precisely? A third question that this result raises is whether outcomes that allocate surplus differently can be robustly achieved. The remainder of the paper addresses the three issues we have raised.

The next subsection presents the assumption on $\boldsymbol{\Theta}$, called recoverable structure, that underlies our analysis of the general robust intervention problem. We then present the key method—a spectral price theory decomposition—that enables our use of this assumption, and illustrate the ideas throughout in relation to the stochastic block model special case.

3.2. **Recoverable structure.** Recoverable structure imposes conditions on the pattern of complements and substitutes (which we will call interactions) among products, as summarized by the Slutsky matrix $\boldsymbol{D}$. It requires that there is a strong latent pattern of product interactions and, simultaneously, this latent structure "has enough correlation" with the vector of market quantities.

We now define this notion formally. A vector in $\mathbb{R}^n$ describes a bundle of products. Given $\boldsymbol{D}$, we are interested in the subspace of bundles spanned by eigenvectors of $\boldsymbol{D}$ with large eigenvalues. Formally, let $\mathcal{L}(\boldsymbol{D}, M) \subseteq \mathbb{R}^n$ be the subspace of the bundle space spanned by the eigenvectors of $\boldsymbol{D}$ with eigenvalues at least M in absolute value.

Definition 2. The set of market states $\boldsymbol{\Theta}$ has (M, δ)-recoverable structure if for every $(\boldsymbol{D}, \boldsymbol{q}^0) \in \boldsymbol{\Theta}$ the projection of $\boldsymbol{q}^0$ onto $\mathcal{L}(\boldsymbol{D}, M)$ has norm at least δ.

To understand this definition, note that a Slutsky matrix, by virtue of being symmetric, can be orthogonally diagonalized: it can be written as a linear combination of orthogonal rank-one matrices:

$$\boldsymbol{D} = -\sum_{\boldsymbol{u}^\ell \in \mathcal{L}(\boldsymbol{D},M)} |\lambda_\ell| \underbrace{\boldsymbol{u}^\ell (\boldsymbol{u}^\ell)^\mathsf{T}}_{\text{rank-1 matrix}} - \sum_{\boldsymbol{u}^\ell \notin \mathcal{L}(\boldsymbol{D},M)} |\lambda_\ell| \underbrace{\boldsymbol{u}^\ell (\boldsymbol{u}^\ell)^\mathsf{T}}_{\text{rank-1 matrix}}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{D}$ (which are nonpositive numbers because $\boldsymbol{D}$ is negative semidefinite), ordered from greatest to least in absolute value, and $(\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n)$ is a corresponding basis of orthonormal eigenvectors. All the summands are orthogonal to each other, and $|\lambda_\ell|$ is the norm of the contribution of the corresponding summand.

[16]When there are $|M|$ types of products, then some type must contain at least $n/|M|$ products.

Having market states with (M, δ)-recoverable structure means that (i) $\boldsymbol{D}$ has eigenvectors with eigenvalues larger than M in absolute value, ensuring the first summation is nonzero, and (ii) the vectors $\boldsymbol{u}^\ell$ in that sum can jointly account for a non-negligible portion of the status quo quantities. We will call the eigenvectors $\boldsymbol{u}^\ell$ associated with the eigenvalues such that $|\lambda_\ell| \geq M$ the *top* eigenvectors.

As we will detail later, if we set M to be larger than the norm of the noise $\boldsymbol{E}$ in the signal of $\boldsymbol{D}$, then condition (i) ensures that the space of top eigenvectors of $\boldsymbol{D}$ can be estimated precisely. We now explain what recoverable structure with a large M means economically before explaining the intuition behind requirement (ii), which we defer to Section 3.3.

3.2.1. *Recoverable structure and large-scale complementarities.* We start with a simple example.

Example 1. Consider the special case where there is only one type of product, i.e., $M = 1$ and $\bar{D}_{k(i),k(j)} = k \in (0, 1)$ and $\gamma(n) = 1$. Then

$$D_{ij} = \begin{cases} -1 & \text{if } i = j, \\ -k & \text{if } i \neq j. \end{cases}.$$

This can be written as

$$\boldsymbol{D} = (k-1)\boldsymbol{I} - k\,\mathbf{1}\mathbf{1}^\top,$$

where $\boldsymbol{I}$ is the identity matrix and $\mathbf{1}$ is the all-ones vector. The eigenvalues of k are $k-1$ with multiplicity $n-1$ and $-k(n-1)-1$ with multiplicity 1. Thus, if $k > M/(n-1)$, and $\sum_i q_i^0 > \delta$, the market state $(\boldsymbol{D}, \boldsymbol{q}^0)$ has (M, δ)-recoverable structure.

Notice that negative off-diagonal entries in $\boldsymbol{D}$ correspond to product complementarities. Thus, even a small (entry-wise) amount of complementarity embedded in the demand system can generate a large eigenvalue and recoverable structure.

Recoverable structure is closely related to large-scale complementarities more generally. To see this, let $\|\boldsymbol{x}\|$ be the Euclidean norm of vector $\boldsymbol{x}$ and note that:

$$|\lambda_1| = \sup_{\dot{\boldsymbol{p}} \neq \boldsymbol{0}} \frac{\|\boldsymbol{D}\dot{\boldsymbol{p}}\|}{\|\dot{\boldsymbol{p}}\|} = \sup_{\dot{\boldsymbol{p}} \neq \boldsymbol{0}} \frac{\|\dot{\boldsymbol{q}}\|}{\|\dot{\boldsymbol{p}}\|},$$

where the first equality is the well-known Courant–Fischer characterization of the spectral radius of a symmetric matrix, and the second equality follows from of equilibrium condition, $\dot{\boldsymbol{q}} = -\boldsymbol{D}\dot{\boldsymbol{p}}$. The solution to the maximization problem over $\dot{\boldsymbol{p}}$ is to choose a change in price equal to the dominant eigenvector of $\boldsymbol{D}$, denoted by $\boldsymbol{u}^1$, which is the eigenvector associated with the largest eigenvalue in absolute value. Thus, the largest eigenvalue measures how much a price shock can be amplified in terms of its effect on demand and the corresponding eigenvector gives this extremal price shock achieving this effect.

This gives us a useful perspective on the meaning of a large $|\lambda_1|$. Note that if there were no demand spillovers across products (i.e., all products were independent), then by our normalization we would have $\boldsymbol{D} = -\boldsymbol{I}$ and each good's quantity would change by an amount equal to the price change. When there are demand spillovers across products, and we change prices in the direction of $\boldsymbol{u}^1$, each good's quantity changes by $-|\lambda_1|$ times the price change, a large negative multiple. Hence, the case $|\lambda_1| > 1$ indicates that there are price changes where the downward effect on quantities is larger than in the independent case. This corresponds to complementarities, in that reductions in demand reinforce one another.[17]

This interpretation of the dominant eigenvector and its eigenvalue extends to the other eigenvectors and eigenvalues of $\boldsymbol{D}$. Let $\boldsymbol{u}^\ell$ be the eigenvector associated with the ℓ^{th} largest eigenvalue, denoted λ_ℓ. A direct implication of the Courant–Fischer theorem is that the price change that maximizes the change in quantities relative to the change in price across all price changes orthogonal to $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^{\ell-1}$ is exactly $\boldsymbol{u}^\ell$, and

$$\sup_{\dot{\boldsymbol{p}} \perp \boldsymbol{u}^x x=1^{\ell-1}} \frac{\|\dot{\boldsymbol{q}}\|}{\|\dot{\boldsymbol{p}}\|} = |\lambda_\ell|.$$

Hence, the spectral decomposition of $\boldsymbol{D}$ captures a set of n orthogonal price changes in the economy, each representing the maximum induced change in quantity that is feasible within the system given the orthogonality constraint.

3.2.2. *Application to the stochastic block model.* We now make explicit the relationship between the general recoverable structure condition and the stochastic block model. In the stochastic block model, fixing $\bar{\boldsymbol{D}}$, $\bar{\boldsymbol{q}}$, and a sequence $\gamma(n)$, the set of market states is the set of all $(\boldsymbol{D}, \boldsymbol{q}^0)$ (for any number of firms) satisfying the block specification (9) for some function k partitioning products into types.

Fact 1. In the stochastic block model, if $\gamma(n)n^{1/2} \to \infty$, then there is a $\delta > 0$ such that, for large enough n, the set of market states has $(M(n), \delta)$-recoverable structure for some constant $\delta > 0$ and a sequence $M(n)$ with $M(n)/n^{1/2} \to \infty$.

The intuition is straightforward: because there are only finitely many types, some blocks in $\boldsymbol{D}$ must be large; as in Example 1, a large block with entries of order $\gamma(n)$ gives rise to an eigenvalue of magnitude $\gamma(n)n$. The proof also shows that the projection onto the corresponding eigenspace of $\boldsymbol{q}^0$ arising from a generic $\bar{\boldsymbol{q}}^0$ has norm bounded below by a number independent of n.

To see how the presence of aggregate structure relates to large-scale complementarities in the example, recall that we assumed $\bar{\boldsymbol{D}}$ itself satisfies NSD, and so its diagonal entries are -1. This means that there is a large block on the diagonal of $\boldsymbol{D}$ with negative entries of magnitude $\gamma(n)$—large-scale complementarities just as in Example 1.

Thus, we can now keep the stochastic block model in mind as a canonical example of recoverable structure with $M(n) \gg \sqrt{n}$.

3.3. **A key tool: Spectral price theory.** When recoverable structure is present, information about the top eigenvectors of $\boldsymbol{D}$ will be recoverable with high precision—under suitable assumptions on errors—using standard statistical results. We now describe how an authority can use information solely about top eigenvectors to design an intervention that increases surplus robustly.

The key tool is a decomposition of surplus pass-through in spectral terms, building on a spectral description of the pass-through of an intervention to

[17]In contrast, when $|\lambda_1| < 1$, the effect on demand is lower than in the independent case, corresponding to an economy where products are globally substitutes.

prices and quantities. Denoting by $\boldsymbol{U}$ the matrix whose ℓ^{th} column is the ℓ^{th} eigenvector $\boldsymbol{u}^\ell$ of $\boldsymbol{D}$, and by $\boldsymbol{\Lambda}$ the matrix whose non-diagonal elements are zero and whose ℓ^{th} diagonal element is λ_ℓ, we have:

$$\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top.$$

An intervention $\boldsymbol{\sigma}$ that subsidizes (or taxes) a single product will in general affect not only the prices and quantities of that product but also those of other products, whose equilibrium values are all connected through strategic interactions. If we think of the eigenvectors $\boldsymbol{u}^\ell$ as representing bundles, then these bundles have the important property that an intervention $\boldsymbol{\sigma} \propto \boldsymbol{u}^\ell$ in the direction of such a bundle will only affect the price $\boldsymbol{u}^\ell \cdot \boldsymbol{p}$ and quantity $\boldsymbol{u}^\ell \cdot \boldsymbol{q}$ of that bundle, leaving the prices and quantities of the bundles corresponding to the other eigenvectors unchanged. Generally, we can decompose $\boldsymbol{\sigma} = \sum_\ell (\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma})\boldsymbol{u}^\ell$ into a combination of n orthogonal interventions, each in the direction of an eigenvector. We can use this decomposition to solve the oligopoly game and obtain simple expressions for the pass-through of the intervention in terms of the eigenvalues of $\boldsymbol{D}$.

Lemma 2. The pass-throughs from any intervention $\boldsymbol{\sigma}$ to prices and quantities of each eigenvector are as follows:

$$\boldsymbol{u}^\ell \cdot \dot{\boldsymbol{p}} = -\frac{1}{1+|\lambda_\ell|}\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{u}^\ell \cdot \dot{\boldsymbol{q}} = \lambda_\ell(\boldsymbol{u}^\ell \cdot \dot{\boldsymbol{p}}) = \frac{|\lambda_\ell|}{1+|\lambda_\ell|}\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}.$$

Proof. From equation (3) we get $\left(\boldsymbol{I} - \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top\right)\dot{\boldsymbol{p}} = \dot{\boldsymbol{c}}$. Multiplying both sides by $\boldsymbol{U}^\top$ we get $\boldsymbol{U}^\top\dot{\boldsymbol{p}} = (\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{U}^\top\dot{\boldsymbol{c}}$ and, using equation (4), $\boldsymbol{U}^\top\dot{\boldsymbol{q}} = \boldsymbol{\Lambda}(\boldsymbol{I} - \boldsymbol{\Lambda})^{-1}\boldsymbol{U}^\top\dot{\boldsymbol{c}}$. □

Thus, we can study the price and quantity pass-throughs of each of these n interventions separately across eigenvectors. In particular, each unit of subsidy in direction $\boldsymbol{u}^\ell$ *exclusively* passes through to the price and quantities of bundle $\boldsymbol{u}^\ell$, and it does so with coefficients $-(1+|\lambda_\ell|)^{-1}$ and $|\lambda_\ell|(1+|\lambda_\ell|)^{-1}$, respectively.

Note that the magnitudes of the price and quantity pass-throughs in the different $\boldsymbol{u}^\ell$ are ordered according to their corresponding eigenvalues: The larger is $|\lambda_\ell|$, the less a given subsidy $\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}$ reduces prices, but the more it increases quantities. This asymmetry is the result of two opposing forces: On the one hand, the strategic interactions among firms imply that the equilibrium price $\boldsymbol{u}^\ell \cdot \boldsymbol{p}^0$ is less sensitive to the subsidy $\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}$ the larger is $|\lambda_\ell|$. On the other, the demand $\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0$ is more sensitive to the price $\boldsymbol{u}^\ell \cdot \boldsymbol{p}^0$ the larger[18] is $|\lambda_\ell|$; this is just a fact about the market's demand function, rather than equilibrium pricing. Lemma 2 shows that the second effect dominates the first in the sense that the larger is $|\lambda_\ell|$, the more sensitive is the equilibrium quantity $\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0$ to the subsidy $\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}$.

3.3.1. *Surplus metrics: Spectral decomposition and robust interventions.* We now combine the spectral decomposition with the surplus formulas to deduce the following second lemma.

[18]Indeed, it follows from (4) that $\boldsymbol{U}^\top\boldsymbol{q}^0 = \boldsymbol{\Lambda}\boldsymbol{U}^\top\boldsymbol{p}^0$, so the slope of the demand $\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0$ with respect to own price $\boldsymbol{u}^\ell \cdot \boldsymbol{p}^0$ is equal to λ_ℓ.

Lemma 3. The pass-throughs to consumer, producer, and total surpluses are:

$$\dot{C} = -\sum_{\ell=1}^{n}(\boldsymbol{u}^{\ell}\cdot\boldsymbol{q}^{0})(\boldsymbol{u}^{\ell}\cdot\dot{\boldsymbol{p}})$$
$$\dot{P} = 2\sum_{\ell=1}^{n}(\boldsymbol{u}^{\ell}\cdot\boldsymbol{q}^{0})(\boldsymbol{u}^{\ell}\cdot\dot{\boldsymbol{q}})$$
$$\dot{W} = \sum_{\ell=1}^{n}(\boldsymbol{u}^{\ell}\cdot\boldsymbol{q}^{0})(\boldsymbol{u}^{\ell}\cdot\dot{\boldsymbol{q}}).$$

Proof. The effect of the intervention on consumer surplus is $\dot{C} = -\boldsymbol{q}^0\cdot\dot{\boldsymbol{p}}$. Multiplying this equation by $\boldsymbol{U}\boldsymbol{U}^{\mathsf{T}}$ gives $\dot{C} = -\boldsymbol{U}^{\mathsf{T}}\boldsymbol{q}^0\cdot\boldsymbol{U}^{\mathsf{T}}\dot{\boldsymbol{p}}$. Similarly, the effect of the intervention on producer surplus is $\dot{P} = \boldsymbol{q}^0\cdot(\dot{\boldsymbol{p}}+\boldsymbol{\sigma}) + (\boldsymbol{p}-\boldsymbol{c})\cdot\dot{\boldsymbol{q}} = 2\boldsymbol{q}^0\cdot\dot{\boldsymbol{q}}$. Multiplying the equation by $\boldsymbol{U}\boldsymbol{U}^{\mathsf{T}}$ yields the expression of $\dot{P}$ in the Lemma. The expression for $\dot{W}$ is obtained by aggregating $\dot{P}$, $\dot{C}$ and the intervention expenditure. □

Lemma 3 shows that the effect of an intervention on consumer, producer, or total surplus is a weighted sum of pass-throughs to each of the eigenvectors $\boldsymbol{u}^{\ell}$—with the weight being the corresponding bundle's quantity.

We now use Lemma 3 to illustrate how—by setting $\boldsymbol{\sigma}$ equal to $\boldsymbol{u}^1$—the authority may achieve the highest total surplus per dollar spent possible subject to the constraint that the change in consumer surplus is not negative (recall Proposition 2). Because $\boldsymbol{u}^1$ is orthogonal to all the other $\boldsymbol{u}^{\ell}$ with $\ell \neq 1$, such intervention changes only the price and quantity of the bundle $\boldsymbol{u}^1$:

$$\boldsymbol{u}^1\cdot\dot{\boldsymbol{p}} = -\frac{1}{1+|\lambda_1|} \quad \text{and} \quad \boldsymbol{u}^1\cdot\dot{\boldsymbol{q}} = \frac{|\lambda_1|}{1+|\lambda_1|},$$

leading to an overall change in consumer and producer surplus equal to

$$\dot{C} = \frac{1}{1+|\lambda_1|}\dot{S} \quad \text{and} \quad \dot{P} = 2\frac{|\lambda_1|}{1+|\lambda_1|}\dot{S}.$$

Recoverable structure requires that $|\lambda_1| \to \infty$ with n, and so (by inspection of the equations) the interventions will achieve $\dot{C}/\dot{S} \to 0$ and $\dot{P}/\dot{S} \to 2$.

4. Conditions for robust surplus improvements

In Proposition 1, we reported a complete information benchmark that shows what is possible with no observation errors. In Proposition 2, we provided an example in which a natural constrained-efficient outcome among these is achievable under considerable observation errors. This section presents our main result, which generalizes this example.

4.1. Assumptions on errors.

We now add to our maintained assumptions the following assumption about the authority's signal. Let $\|\boldsymbol{E}\|$ denote the spectral norm of a matrix $\boldsymbol{E}$, which for a symmetric matrix is equal to its largest eigenvalue. Fix a sequence $b(n)$ and a positive constant $\overline{V}$.

Assumption 4. We assume that:

(1) $\mathbb{E}[\|\boldsymbol{E}\|] \leq b(n)$;

(2) all the ε_i are independent and $\mathbb{E}[\|\boldsymbol{\varepsilon}\|^2] \leq \overline{V}$.

The first part of the assumption bounds the matrix norm of the errors in estimating the normalized Slutsky matrix. In Appendix C, we provide a simple procedure for sampling market data independently across product pairs (i, j) under which this assumption holds with $b(n) = \gamma n^{1/2}$ (for a positive constant $\gamma > 0$). We can think of this example as generating E_{ij} that are essentially independent, with variance of constant order (i.e., neither growing nor decaying with n). This shows that the assumption can hold even when there is no entry of $\boldsymbol{D}$ that can be recovered accurately.

If $b(n) = \gamma n^{\beta}$ with $\beta \in (1/2, 1)$, there are error structures consistent with Assumption 4 where some terms in $\widehat{\boldsymbol{D}}$ have large covariances. For example, spatially correlated errors, with sufficiently "distant" parts of the matrix being at most slightly correlated, would satisfy this assumption.[19] An example that would violate part (1) of Assumption 4 is the entries of $\boldsymbol{E}$ all having correlation bounded away from zero (e.g., arising from a common shock to measured complementarities).

The second part of the assumption requires independence of errors across different products' quantities. The purpose of the assumption is to apply a law of large numbers for estimating outcomes such as the average quantity, as well as various linear combinations of quantities that are important for intervention outcomes. Independence is stronger than we need for our main result stated in Theorem 1, and is made to facilitate exposition. When we use this assumption in the proof of our main result, Theorem 1, we rely on a substantially weaker but more technical condition that Assumption 4(2) implies (see Appendix A.3). Regarding the assumption on the norm of $\boldsymbol{\varepsilon}$, recall that we assume that $\|\boldsymbol{q}^0\| \leq 1$; the assumption on $\boldsymbol{\varepsilon}$ scales the error to be of the same order of magnitude as the quantity vector.

4.2. **Main result.** Our main result, Theorem 1, examines what can be implemented robustly when the authority has partial information and the economy has a recoverable structure that is strong enough relative to the noise. It shows that interventions exist that robustly protect consumers from surplus loss and implement market surplus equal to the upper bound achievable by an omniscient authority (the upper bound given by Proposition 1(2)).

Recall that we have fixed a sequence $b(n)$ under which Assumption 4 holds—an upper bound on the noise in observations of the demand system.

Theorem 1. Let $M(n)$ be an increasing sequence with $M(n)/b(n) \to_n \infty$, and fix $\delta > 0$. Assume the set of market states has $(M(n), \delta)$-recoverable structure. For every $\epsilon > 0$ and for every target expenditure $s > 0$, the following properties can be simultaneously achieved ϵ–robustly for sufficiently large n:

(i) The sum of marginal consumer and producer surplus gains $\dot{C} + \dot{P}$ is at least twice the marginal expenditure, up to a small multiplicative error: $\dot{C} + \dot{P} \geq (2 - \epsilon)\dot{S}$.

(ii) The marginal effect on consumer surplus is not significantly negative: $\dot{C} \geq -\epsilon$. Moreover, no individual consumer's surplus decreases significantly—i.e., $\dot{C}^h \geq -\epsilon$ for each household h.

[19]The idea is analogous to ergodicity-type conditions in time-series settings.

(iii) The marginal expenditure is arbitrarily close to the target expenditure s, i.e., $|\dot{S} - s| < \epsilon$.

The condition in Theorem 1 stipulates that the market has $(M(n), \delta)$-recoverable structure for some $M(n)$ that asymptotically dominates $b(n)$. This lower bound on $M(n)$ ensures that the recoverable structure in $\boldsymbol{D}$ is substantially larger than the norm of the error matrix $\boldsymbol{E}$, which is $O(b(n))$ under Assumption 4. Note also that the assumptions on errors are satisfied by the stochastic block model of Section 3.1, so Proposition 2 is a direct corollary of Theorem 1.[20]

Part (i) states that, when this condition is satisfied, the authority can robustly achieve approximately two dollars of surplus gain per dollar spent.[21]

Point (ii) states that it is possible to achieve this while leaving all households' welfare essentially unchanged; indeed, under the policy we construct, producers fully capture the surplus gains. Note that by Proposition 1, the welfare gain described in point (i) is essentially the maximum total surplus change that an omniscient authority could implement with the same expenditure without reducing consumer surplus.

Finally, point (iii) says the authority can precisely target the realized expenditure (and thus, total surplus impact) of the policy.

Our notion of ϵ–robustness means that these surplus properties are achieved ex post with high probability. A natural question is whether they are also achieved in expectation (since, in principle, realizations with very negative surplus could occur with low probability). In our setting, it turns out that all the analysis would be unaffected if we added good ex ante expected performance to the definition of robustness.[22]

To prove Theorem 1, we apply a statistical method that accurately identifies a subspace spanned by top eigenvectors of the Slutsky matrix from noisy observations. We then show that interventions projecting exclusively onto these recoverable subspaces possess the desirable welfare properties stated in Theorem 1.

4.3. **Recovering and using the subspace of top eigenvectors.**

4.3.1. *The Davis–Kahan theorem.* Recall that if $M(n) \gg b(n)$, then $(M(n), \delta)$-recoverable structure requires that the normalized Slutsky matrix $\boldsymbol{D}$ has eigenvalues that, in absolute value, are much larger than $b(n)$; we will refer to such eigenvalues simply as "large" from now on.

The key tool in our statistical exercise that leverages this assumption is the Davis–Kahan theorem. Under the hypothesis that some eigenvalues of $\boldsymbol{D}$ are large, this theorem guarantees that, despite the noise in $\boldsymbol{E}$, the large eigenvalues of the observed matrix are good approximations of the true large eigenvalues. In other words, the noise in $\boldsymbol{E}$ cannot cause the large eigenvalues of $\boldsymbol{D}$ to become "mixed up" with the eigenvalues far away in the spectrum; see Figure 1

[20] This follows since the entries of $\boldsymbol{E}$ were taken to be independent (Dallaporta, 2012).

[21] Recalling the definition $\dot{W} = \dot{P} + \dot{C} - \dot{S}$, this implies that every dollar spent yields approximately one unit increase in $\dot{W}$, net total surplus.

[22] This is because the surplus pass-throughs are supported on $[0, 1]$ and all quantities in the proofs are bounded. Thus, convergence in probability is equivalent to convergence in L^1, and so our proofs extend to show close approximations to the omniscient benchmark in terms of ex ante surplus.

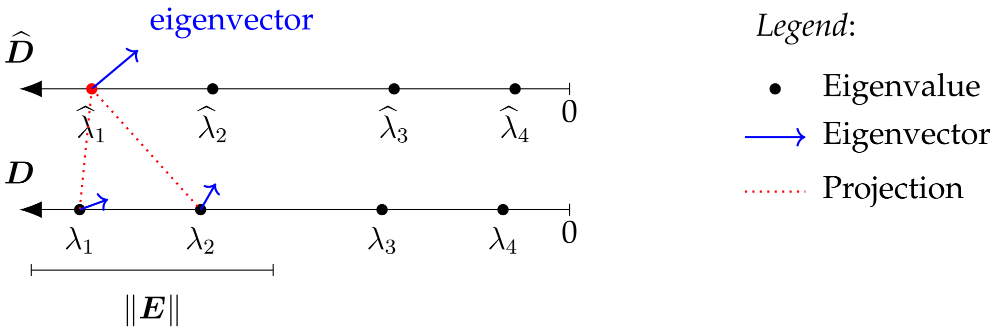


FIGURE 1. An illustration of the Davis–Kahan theorem: How eigenvectors of $\widehat{\boldsymbol{D}}$ (perturbed matrix) project onto eigenvectors of $\boldsymbol{D}$ (true matrix) with similar eigenvalues (off by at most $\|\boldsymbol{E}\|$). This relationship ensures that the subspace generated by the "top" (i.e., large-eigenvalue) eigenvectors of $\widehat{\boldsymbol{D}}$ is a good approximation of the subspace generated by the top eigenvectors of $\boldsymbol{D}$. For our economic problem this implies that interventions based on top eigenvectors of $\widehat{\boldsymbol{D}}$ yield high surplus pass-through, despite noise $\boldsymbol{E}$.

for an illustration. The theorem also permits the recovery of eigenvectors. More precisely, this theorem has the following two central implications in our setting:

(i) $\widehat{\boldsymbol{D}} = \boldsymbol{D} + \boldsymbol{E}$ has some eigenvalues which are themselves large;
(ii) the eigenvectors of $\widehat{\boldsymbol{D}}$ associated with such eigenvalues are in the span of eigenvectors of $\boldsymbol{D}$ with large eigenvalues, i.e., the eigenvectors of $\widehat{\boldsymbol{D}}$ associated with such eigenvalues can be expressed (up to a small error) as linear combinations of eigenvectors of $\boldsymbol{D}$ with large eigenvalues.

4.3.2. *Using recovery of eigenvectors to design a subsidy policy: A simple illustration.* This powerful result forms the core of our strategy to recover and use structure underlying the oligopoly demand robustly under noise. To facilitate the illustration, we impose a stronger assumption on $\boldsymbol{D}$: that the largest eigenvalue of $\boldsymbol{D}$ is sufficiently well-separated from all other eigenvalues by a "gap" much larger than $b(n)$. Under this condition, the Davis–Kahan Theorem yields an even stronger implication: we can use $\widehat{\boldsymbol{D}}$ to recover a normalized eigenvector $\widehat{\boldsymbol{u}}^1$ that correlates almost perfectly with the corresponding eigenvector[23] $\boldsymbol{u}^1$.

To use this, observe that Lemma 2 and Lemma 3 together imply

$$\dot{W} = \sum_{\ell=1}^{n} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) \frac{|\lambda_\ell|}{1 + |\lambda_\ell|}, \tag{10}$$

while expenditure is

$$\dot{S} = \sum_{\ell=1}^{n} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}). \tag{11}$$

Let us use the recovered $\widehat{\boldsymbol{u}}^1$ to design an intervention with $\boldsymbol{\sigma} \propto \widehat{\boldsymbol{u}}^1$. The Davis–Kahan Theorem allows us to treat $\widehat{\boldsymbol{u}}^1$ as effectively equal to its true counterpart $\boldsymbol{u}^1$ with a very small error, so from now on we will treat $\boldsymbol{u}^1$ as known. By choosing the sign of $\boldsymbol{\sigma}$ to ensure $(\boldsymbol{u}^1 \cdot \boldsymbol{q}^0)(\boldsymbol{u}^1 \cdot \boldsymbol{\sigma})$ is positive. Then we can see

[23] In Section 5 we develop Example 2 to illustrate our main result. Figure 2 illustrates the similarity between the true $\boldsymbol{u}^1$ (in panel B) and the estimated $\widehat{\boldsymbol{u}}^1$ (panel D) for a case where this stronger "gap" condition holds.

from the equations above that $\dot{W}$ will closely approximate $\dot{S}$, since $\lambda_1 \gg b(n)$ implies $\frac{|\lambda_1|}{1+|\lambda_1|} \approx 1$. Moreover, if we know $\boldsymbol{u}^1 \cdot \boldsymbol{q}^0$ and this differs from zero (which is a requirement of recoverable structure), we can scale the intervention to be of the size that we desire, and achieve $\dot{S} = s$.

This argument contains some wishful thinking, however. When we arranged the sign of $\boldsymbol{\sigma}$ so that $(\boldsymbol{u}^1 \cdot \boldsymbol{q}^0)(\boldsymbol{u}^1 \cdot \boldsymbol{\sigma})$ is positive, we did not consider that we have only a noisy observation $\widehat{\boldsymbol{q}}^0$ of $\boldsymbol{q}^0$. So part of the challenge of the proof is to manage the observation error that makes $\widehat{\boldsymbol{q}}^0$ different from $\boldsymbol{q}^0$, and to show that we can obtain a correct estimate of the sign with probability tending to 1 as $n \to \infty$. If we fail to do this correctly, our intervention actually decreases efficiency with positive probability. This explains the need for the second part of Assumption 4 on the error $\boldsymbol{\varepsilon}$ in the quantity signal.

However, bounded noise alone is insufficient: if $\boldsymbol{u}^1 \cdot \boldsymbol{q}^0$ is very small, there may be no hope for consistently recovering the true magnitude or sign of $\boldsymbol{u}^1 \cdot \boldsymbol{q}^0$ from the signal $\widehat{\boldsymbol{u}}^1 \cdot (\boldsymbol{q}^0 + \boldsymbol{\varepsilon})$ even with well-behaved noise: the asymptotically small noise could still overwhelm a similarly decaying underlying mean $\boldsymbol{u}^1 \cdot \boldsymbol{q}^0$. Such an unrecoverability would make it impossible to orient and scale our intervention appropriately. The definition of recoverable structure prevents this problem by requiring that the projection of $\boldsymbol{q}^0$ onto eigenvectors with large eigenvalues is bounded away from zero.

The special case of our main result that this discussion makes plausible is: If the largest eigenvalue of $\boldsymbol{D}$ is well-separated from others and if $\boldsymbol{q}^0 \cdot \boldsymbol{u}^1$ is not vanishingly small, then a subsidy profile proportional to $\boldsymbol{u}^1$ can, if it is suitably scaled, achieve all the properties of Theorem 1.

4.3.3. *The more general result.* The proof of the main result improves on this sketch in two ways. First, it does not rely only on the eigenspace spanned by $\boldsymbol{u}^1$. Instead, it uses a potentially much larger eigenspace of $\widehat{\boldsymbol{D}}$. The general intervention projects $\widehat{\boldsymbol{q}}^0$ onto $\mathcal{L}(\boldsymbol{D}, M(n))$, the eigenspace of all eigenvectors of $\boldsymbol{D}$ with eigenvalues larger than $M(n)$. This makes it easier for the analog of $\boldsymbol{u}^1 \cdot \boldsymbol{q}^0$ not to be too small, since the projection of $\boldsymbol{q}^0$ onto a larger eigenspace will have a larger norm. Second, the general proof dispenses with assuming that any eigenvalues are well-separated. Instead, it handles any possible spectrum of $\boldsymbol{D}$ subject to our maintained assumptions. This introduces considerable complexity, as it is no longer generally possible to recover any true eigenvector $\boldsymbol{u}^\ell$ with any accuracy. We instead work directly with a recovered eigenspace that generalizes the span of $\widehat{\boldsymbol{u}}^1$. We show that despite limited knowledge of individual true eigenvectors of $\boldsymbol{D}$ underlying this space, we can use the fact that all of them have large eigenvalues to generalize our argument for showing that (10) and (11) can be made very close and nonzero with a feasible intervention. This is where the arguments go beyond standard applications of the Davis–Kahan theorem.

5. Illustration: Interventions based on noisy demand estimates

We begin this section by providing an example illustrating the notion of recoverable structure, and then use this example to demonstrate the effects of the intervention rule that taxes and subsidizes firms in the direction of the eigenvector associated with the largest eigenvalue of the observed Slutsky matrix.

Example 2 (An illustration of recoverable structure)**.** There are $n = 300$ products. The Slutsky matrix is the combination of two matrices:

$$\boldsymbol{D} = (1-\gamma)\boldsymbol{D}_{\text{block}} + \gamma(-\boldsymbol{Z}\boldsymbol{Z}'). \tag{12}$$

The first matrix, $\boldsymbol{D}_{\text{block}}$, is a block matrix dividing the 300 products into 3 equally-sized blocks. The entries of $\boldsymbol{D}_{\text{block}}$ are based on a 3×3 matrix $\boldsymbol{C}$ that governs the pattern of interactions within blocks (C_{ii}) and across them (C_{ij} with $i \neq j$). In particular,[24]

$$\boldsymbol{C} = \begin{pmatrix} -1 & 0.15 & 0.7 \\ 0.15 & -1 & 0.6 \\ 0.7 & 0.6 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{D}_{\text{block}} = \boldsymbol{C} \otimes \boldsymbol{J}_{n/3}.$$

Here $\boldsymbol{J}$ is the matrix of ones. Products within each block are complements, while products across blocks are substitutes. For instance, products in block 3 are highly substitutable with products in other blocks, whereas products in blocks 1 and 2 are only mildly substitutable with each other.

We can think of each block as a product category—for example, kitchen products, digital entertainment products, and sports-related products. Within categories, products are complements, but across the categories they are substitutes. For another example, consider three non-compatible operating systems accomplishing similar tasks; products within each system are complements, and products across them are substitutes.

The second term in eq. (12), a scaling of $-\boldsymbol{Z}\boldsymbol{Z}'$, is a negative definite matrix that adds heterogeneity to the regular pattern of interactions in $\boldsymbol{D}_{\text{block}}$. In particular, we take $\boldsymbol{Z}$ to be an $n \times 10$ matrix with rows drawn uniformly from the unit sphere.[25]

Finally, the observed Slutsky matrix is

$$\widehat{\boldsymbol{D}} = \boldsymbol{D} + \boldsymbol{E}$$

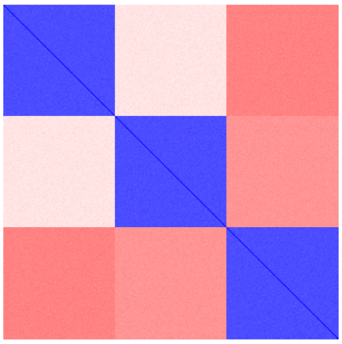
where each entry of the error matrix $\boldsymbol{E}$—except its diagonal entries, which are kept at zero—is drawn from $U[-1, 1]$; this noise structure results in the upper bound $b(n) = n^{1/2}$ on $\|\boldsymbol{E}\|$.

We aim to illustrate basic statistical implications of (M, δ)-recoverable structure that is strong enough relative to the noise $b(n)$—that is, with M much larger than $b(n)$. In this example, the largest eigenvalue of $\boldsymbol{D}_{\text{block}}$ substantially exceeds $n^{1/2}$. Hence, if γ is low, then the $\boldsymbol{D}$ of eq. (12) will also have some eigenvalues exceeding $n^{1/2}$. In contrast, if γ is high, then the main contributor to $\boldsymbol{D}$ is $-\boldsymbol{Z}\boldsymbol{Z}'$ and, in this case, the largest eigenvalues of $\boldsymbol{D}$ turn out to be no larger than $n^{1/2}$. We now show that these two cases, low vs. high γ, have very different implications for inference.

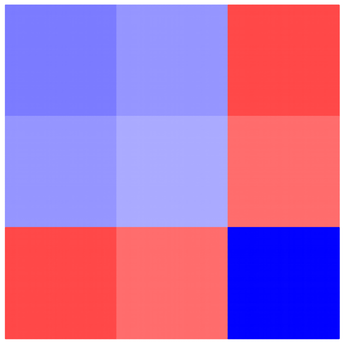
First we consider a low value of $\gamma = 0.3$. In this case, the two largest eigenvalues of $\boldsymbol{D}$, in absolute value, are approximately 130 and 80, and these are considerably larger than $b(n) = \sqrt{300} \approx 17$, whereas the third largest eigenvalue is roughly 1.2; hence, $\mathcal{L}(\boldsymbol{D}, M)$ with $M > b(n)$ equal to, e.g., $n^{2/3}$, is the subspace spanned by eigenvectors $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$. As we explained in Section 4.3, the Davis-Kahan Theorem tells us that we can use the noisy observation $\widehat{\boldsymbol{D}}$ to

[24]The symbol $\otimes$ denotes the Kronecker product.

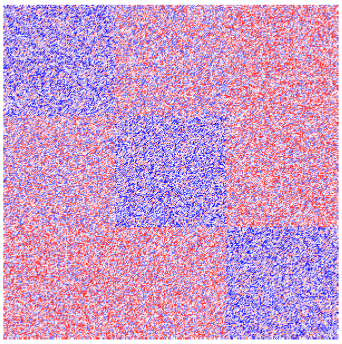
[25]Note that entry (i, i) of $\boldsymbol{Z}\boldsymbol{Z}'$ is the norm of the ith row of Z, which, by construction, is 1.



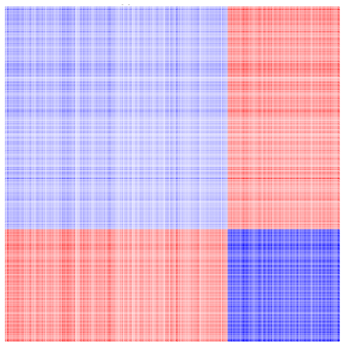
(A) True Slutsky matrix $\boldsymbol{D}$



(B) True latent structure: $\boldsymbol{u}^1(\boldsymbol{u}^1)^\mathsf{T}$



(C) Slutsky matrix signal $\widehat{\boldsymbol{D}}$



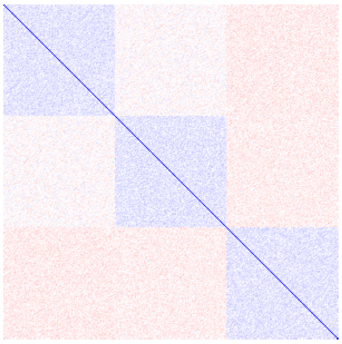
(D) Estimated latent structure: $\widehat{\boldsymbol{u}}^1(\widehat{\boldsymbol{u}}^1)^\mathsf{T}$

FIGURE 2. Illustration of true vs. estimated parameters when there is recoverable structure, $\gamma = 0.3$. Blue regions illustrate complementarities (a negative sign of the corresponding matrix entry) and the red regions signify substitutabilities (positive signs). The darker the color, the higher the corresponding entry in absolute value.

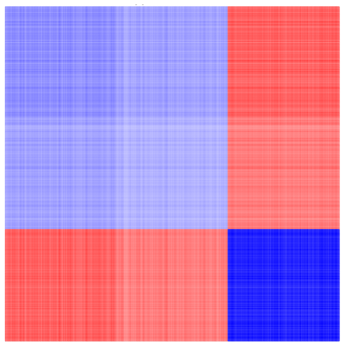
accurately estimate the eigenvectors in $\mathcal{L}(\boldsymbol{D}, M)$. Here, for simplicity, we illustrate graphically how the first eigenvector of $\widehat{\boldsymbol{D}}$ is a good description of the first eigenvector of $\boldsymbol{D}$.

Figure 2 depicts the Slutsky matrix when $\gamma = 0.3$. Panel A illustrates the true matrix $\boldsymbol{D}$ and panel B illustrates the rank-1 matrix $\boldsymbol{u}^1(\boldsymbol{u}^1)^\mathsf{T}$ corresponding to the eigenvector with the highest eigenvalue. We can see that the rank-1 matrix of Panel B, despite being defined based on how $\boldsymbol{D}$ acts on a very low-dimensional subspace, captures a lot of information about the pattern of demand interactions within and across blocks of $\boldsymbol{D}$. Panel C illustrates a realization of the observed matrix $\widehat{\boldsymbol{D}}$. We now illustrate how the patterns in $\boldsymbol{D}$ can be recovered from this observation. Let $\widehat{\boldsymbol{u}}^1$ be the eigenvector of $\widehat{\boldsymbol{D}}$ associated with its largest eigenvalue. Panel D illustrates the associated rank-1 matrix $\widehat{\boldsymbol{u}}^1(\widehat{\boldsymbol{u}}^1)^\mathsf{T}$. The close resemblance between the matrices in Panel D and Panel B is an informal illustration of how $\widehat{\boldsymbol{u}}^1$ is a good approximation of the true $\boldsymbol{u}^1$. That is, despite the fact that we observe $\boldsymbol{D}$ with substantial noise, we can recover structure latent in $\boldsymbol{D}$, summarized by $\boldsymbol{u}^1(\boldsymbol{u}^1)^\mathsf{T}$, by simply calculating the largest-eigenvalue eigenvector of the noisy matrix $\widehat{\boldsymbol{D}}$.

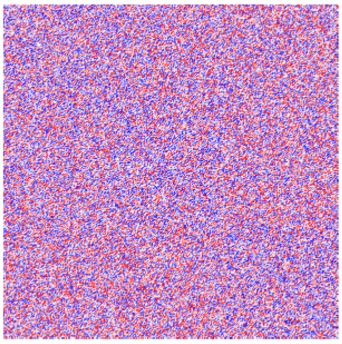
Figure 3 illustrates the Slutsky matrix when $\gamma = 0.9$. In this case the largest eigenvalue is comparable to $\sqrt{n}$ and therefore is not much larger than the noise. Although $\gamma = 0.9$, and so $\boldsymbol{D}$ mainly consists of $-\boldsymbol{Z}\boldsymbol{Z}'$, the block matrix $\boldsymbol{D}_{\text{block}}$ is still visible in $\boldsymbol{D}$ (Panel A) and the true rank-1 matrix $\boldsymbol{u}^1(\boldsymbol{u}^1)^\intercal$ summarizes that pattern (Panel B). However, this structure cannot be recovered using noisy observation. This is illustrated by a realization of the estimated matrix $\widehat{\boldsymbol{D}}$ (Panel C) and the associated rank-1 matrix $\widehat{\boldsymbol{u}}^1(\widehat{\boldsymbol{u}}^1)^\intercal$ (Panel D).



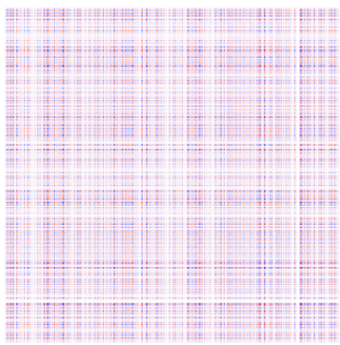
(A) True Slutsky matrix $\boldsymbol{D}$



(B) True latent structure: $\boldsymbol{u}^1(\boldsymbol{u}^1)^\intercal$



(C) Slutsky matrix signal $\widehat{\boldsymbol{D}}$



(D) Estimated latent structure: $\widehat{\boldsymbol{u}}^1(\widehat{\boldsymbol{u}}^1)^\intercal$

FIGURE 3. Illustration of true vs. estimated parameters when there is not recoverable structure, $\gamma = 0.9$. The meaning of the colors is as in Figure 3c

5.1. **Interventions.** In the context of Example 2, we focus on the following intervention rule: Recover the eigenvector associated to the largest eigenvalue, in absolute terms, of the estimated Slutsky matrix $\widehat{\boldsymbol{D}}$. Intervene to subsidize firms in proportion to this eigenvector. This intervention aligns with the intervention rule behind our Theorem 1.[26]

In order to meaningfully compare the effects of such intervention for different realizations of the market state, we scale the size of all interventions that

[26] In this example, for low values of γ the largest eigenvalue of $\boldsymbol{D}$ is sufficiently well-separated from all other eigenvalues and, consequently, the authority can use the first eigenvector of $\widehat{\boldsymbol{D}}$ as a good approximation of $\boldsymbol{u}^1$ (see the illustrative discussion in Section 4.3.2).

we consider by requiring that they have the same expenditure based on the observed quantity vector.[27]

The true initial quantity vector $\boldsymbol{q}^0$ has some regular block structure but also some idiosyncratic heterogeneity. It is constructed as follows:

$$q_i^0 = (\boldsymbol{q}_{\text{block}})_i X_i$$

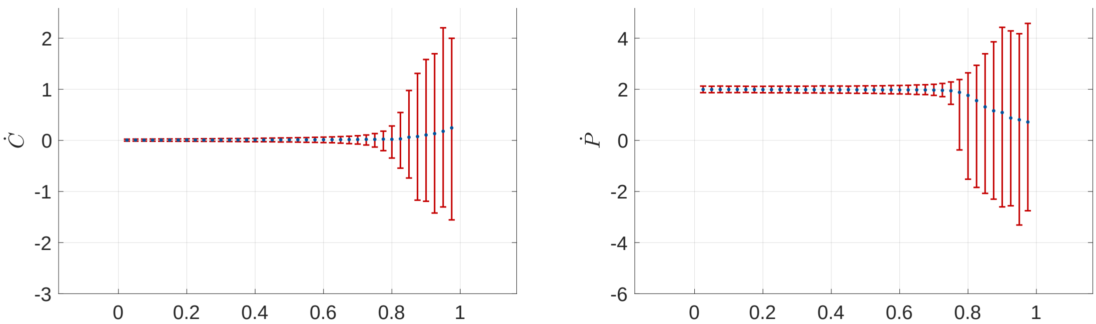
Here, the quantity vector $\boldsymbol{q}_{\text{block}}$ provides a base quantity for each product that depends on its associated block (0.1 for products in the first two blocks, and 3 for the products in the third block). The random variable X_i is drawn independently of all others, and its logarithm is normal with variance 0.1 and mean 1. We use a multiplicative perturbation to avoid negative quantities.

The observed quantities are given by

$$\widehat{q}_i^0 = q_i^0 Y_i$$

where Y_i is an independent error with the same distribution as X_i. This can be rewritten in terms of our additive error model, with $\varepsilon_i = q_i^0(Y_i - 1)$. Here again, the multiplicative error model avoids negative quantities.

5.2. **Evaluation of interventions.** We consider different values of $\gamma \in [0, 1]$. For each of these values we generate 3000 market states according to the above description and we compute the changes in consumer and producer surplus under the true market state. Figure 4 summarizes this exercise: for each value of γ considered, it reports the median (blue dot) of the change in consumer surplus (panel A) and of the change in producer surplus (panel B) and the respective 5th and 95th percentiles associated with the 3000 market state realizations.



(A) Change in consumer surplus

(B) Change in producer surplus

FIGURE 4. The effects of a first eigenvector intervention on changes in consumer surplus (panel a) and changes in producer surplus (panel b) as a function of γ in a market described in Example 2. For each value of γ, we plot the median (in blue), 5th, and 95th percentiles associated with 3000 market state realizations.

Figure 4 shows a sharp transition in the performance of the intervention. For γ less than roughly 0.7 the true market state has very large eigenvalues and so the authority can use the estimate of $\boldsymbol{D}$ to precisely identify the underlying main eigenvector(see Figure 2 for $\gamma = 0.3$.) Note also that products in category 3 are the ones that are most substitutable with other products in categories 1 and 2 and so they are highly represented in the first eigenvector (i.e., they have

[27]More precisely, we first project the observed quantity vector onto the recovered eigenvector, and use that to predict the expenditure size.

a high eigenvector centrality). This implies that the estimated first eigenvector is sufficiently correlated with the status quo market quantity. Hence, for low γ, the true market state has recoverable structure. This allows the authority to implement interventions that robustly have the pass-through properties characteristic of high-eigenvalue eigenvectors—negligible impact on prices and hence on consumers, along with an effect on producer surplus equal to twice the authority's spending , which is normalized to one unit.

However, as γ grows larger than 0.7, the property of recoverable structure fails (see Figure 3 for $\gamma = 0.9$) with the consequence that an intervention that taxes and subsidizes firms based on the estimated first eigenvector is very unpredictable and risky. The unpredictability is shown by the fast widening of the error bars as γ increases beyond 0.7. The riskiness is shown by the fact that for over a third of the outcomes, the realized change in producer surplus, and hence in total surplus, is negative.

In this example, the demand for products in block 3 is significantly higher than the demand for the other products. Hence, to a first approximation, consumer surplus increases when the price of products in block 3 decreases, while producer and total surplus increase when the quantity of these products increases.

When the authority can estimate the first eigenvector accurately, the first eigenvector intervention turns out to subsidize products in block 3 and tax all the other products. Subsidizing products in block 3 leads to a decrease in their price and hence an increase in their demand. Taxing products in blocks 1 and 2 leads to an increase of the price of these products, and hence an increase in the demand of products in block 3. Combining the effects, the intervention leads to a relatively high increase in the demand of good 3, while keeping prices roughly constant. As a result, both producer and total surplus increase dramatically without sizable changes in consumer surplus.

This management of the spillovers can be achieved only by statistically identifying some relevant latent market structure from the noisy demand measurements. While in this example that structure takes the simple form of product categories, in general it might be less easy to describe, and yet equally useful for the design of robust interventions.

6. Tightness of the main result

We have established that if demand has recoverable structure, the authority can robustly achieve the maximum possible total surplus per dollar spent subject to the constraint that consumers are not harmed. The associated intervention rule boosts production with minimal price changes, resulting in firms capturing all efficiency gains.

This raises two natural questions. First, can we find interventions that robustly increase total surplus when the demand does not have recoverable structure? Second, when there is recoverable structure, is the structure of robust interventions that we have just described in any sense necessary? For example, could we have found interventions that robustly increase consumer surplus, rather than leaving it unchanged?

In this section, we work with the case we have focused on in other illustrations, where $\boldsymbol{E}$ has i.i.d. entries with a standard deviation that does not depend on n, giving $b(n) \sim n^{1/2}$; this choice is immaterial and the results apply to a wide range of alternative noise structures.

Proposition 3. The following hold:

1. There are environments such that (i) for any $\epsilon > 0$, there are no intervention rules that ϵ–robustly increase total surplus ($\dot{W} > 0$) for any n; (ii) there are interventions achieving $\dot{P} = 2\dot{S}$ and $\dot{C} = 0$ for any $\dot{S} \geq 0$.
2. There are environments satisfying $(M(n), \delta)$-recoverable structure with $\frac{M(n)}{b(n)} \to \infty$ and $\delta > 0$ such that (i) for any $\epsilon > 0$, there are no intervention rules that ϵ–robustly achieve $\dot{C} > \epsilon$ for all n; (ii) there are interventions achieving $\dot{C} = \dot{S}$ and $\dot{P} = 0$ for any $\dot{S} \geq 0$.

In each case, Proposition 3 describes the limits on what can be achieved by an authority with noisy information. It also notes that these limits really are about information: part (ii) of each case states that an omniscient authority would not be subject to the same limitation.

In more detail, Part 1 of Proposition 3 tells us that we may not be able to design interventions that robustly increase total surplus $\dot{W} = \dot{C} + \dot{P} - \dot{S}$. (Since, by Theorem 1 we know this can be done under the recoverable structure assumption, our construction must lack recoverable structure.) Intuitively, in this case the information about the market state learned from the signal can be very imprecise and, therefore, there are market states in which any intervention will lead to undesirable outcomes. The proof constructs a set of market states such that, with an uninformative signal, for any intervention there is a market state with $\dot{W} < 0$. The basic idea is to use the total surplus decomposition:

$$\dot{W} = \sum_{\ell} (\boldsymbol{u}^{\ell} \cdot \boldsymbol{q}^{0})(\boldsymbol{u}^{\ell} \cdot \boldsymbol{\sigma}) \frac{|\lambda_\ell|}{1 + |\lambda_\ell|} \tag{13}$$

and construct the example so that the authority cannot accurately predict the signs of the terms for any given $\boldsymbol{\sigma}$.

Part 2 of Proposition 3 tells us that even if the market state has recoverable structure, it may be impossible to design interventions that robustly increase total surplus *and* allow consumers to capture some of the resulting efficiency gains. Lemma 3 tells us that to achieve such an outcome, the intervention must project onto some $\boldsymbol{u}^{\ell}$ where λ_ℓ is not too large, since only those eigenvectors have nonvanishing pass-through to consumer surplus. However, the noisy observation of $\widehat{\boldsymbol{D}}$ and $\boldsymbol{q}^0$ gives very noisy estimates of the constituents of (13) corresponding to these eigenvectors. So, by targeting them, there is a substantial chance (at least in some environments) that the policy will have negative consequences for consumers.

7. Discussion and concluding remarks

We have developed a theory of robust interventions in large oligopolies. We identify a condition on demand under which an authority can robustly increase the total surplus per dollar spent as much as would be possible under perfect information subject to the constraint that consumers are not harmed.

The methodological contribution lies in developing spectral methods to analyze pass-through in oligopolies, and applying these methods to gain leverage on statistical problems about oligopolies observed with noise.

We conclude with some observations about the scope of our analysis and connections to related research.

7.1. **Marketplace big data in practice.** In our model, the authority has "big data" about the Slutsky matrix $\boldsymbol{D}$ that may not allow precise estimation of any pairwise demand interactions or hedonic model parameters. This is natural for markets with a large and changing collection of goods, such as those hosted on large online marketplaces. Such marketplaces collect immense amounts of data—about consumer browsing behavior, timing of purchases, consideration sets, etc.—and apply machine learning techniques to these data to form informative but imperfect estimates of interactions among various goods (Athey, 2018; Wager and Xu, 2021; Cai and Daskalakis, 2022; Bajari, Burdick, Imbens, Masoero, McQueen, Richardson, and Rosen, 2023). This is modeled by our notion of a market signal.[28] Our paper offers an approach for calculating suitable statistics that suffice for effective interventions despite the noise in this signal.

Our approach contrasts with a standard one in empirical industrial organization, where markets tend to be defined tightly so that each contains only a relatively small number of similar goods with strong demand interactions, and then a small number of hedonic parameters and demand elasticities are precisely estimated. That approach would correspond in our notation to a very precise signal (i.e., an error matrix $\boldsymbol{E}$ with small norm) and a small number of products.

We have focused on total market surplus as a canonical objective. But we also saw that our robust intervention maximizes the change in total surplus by increasing producer surplus while holding consumer surplus constant. In this sense, the intervention maximizes the increase in producer surplus under the constraint that consumers do not lose—a reasonable objective for an operator of a marketplace that collects revenue proportional to sellers' profits. We leave the study to other objective functions for the authority to future research.

Our model permits flexible marginal interventions. This modeling choice is suited to online marketplaces, because the operators in charge of them can finely target policies that function as taxes and subsidies, including commission rates, discount coupons, free advertising, etc.—and regularly experiment with such perturbations. It is worth noting, however, that the policies our analysis recommends need not be specific to individual products. This is because when we take a large matrix (in our case, the Slutsky matrix) reflecting relationships among units (in our case, products) and look at eigenvectors with large eigenvalues, the coordinates of those eigenvectors typically yield low-dimensional embeddings capturing substantively natural categories (Chen, Chi, Fan, and Ma, 2021). For instance, in our Example 1, the top two eigenvectors are sufficient to recover the blocks to which the goods belong. Relatedly, *spectral clustering* analyses based on the top few eigenvectors sort items into natural "similarity" classes, where similarity is defined by having similar relationships to

[28]The distributional properties of $\boldsymbol{E}$ describing the errors in these estimates would depend on the application, as would the conditions for $\boldsymbol{E}$ that would bound its norm. It would be interesting to investigate these issues in specific applications.

other classes (Spielman and Teng, 1996). Once again, the spectral statistics used in these techniques tend to pick up interpretable "broad" features of the products, rather than idiosyncrasies specific to individual products. As a result, the policies our interventions recommend—which project all variation onto these eigenvectors—will often be close to a policy that depends mostly on category—e.g., a subsidy on smartphones along with a tax on certain types of accessories. Though the policies will not be perfectly regular (note the irregularities of Panel D of Figure 2) the above observations lead us to conjecture that an authority constrained to design policies that discriminate only at a coarse product level could, under natural assumptions, achieve a substantial amount of the gains of our policies. We leave these interesting considerations to future work.

Lastly, we mention the problem of predicting the effects of general perturbations to markets in settings such as ours. This is related to, but distinct from, the problem we have studied. The problem of robust intervention is importantly easier, because the authority chooses the perturbation to make a prediction about. Nevertheless, the spectral decomposition of intervention effects appears likely to be useful for descriptive comparative statics in cases where a cost shock $\boldsymbol{\sigma}$ or other change is exogenous.

7.2. **Relationship with hedonic utility models.** Recent work by Pellegrino (2021) and Ederer and Pellegrino (2021) uses an oligopoly model to empirically quantify the evolution of market power. The relationship between our model and their work sheds light on the types of empirical models that can capture recoverable structure.

In Pellegrino (2021), the model of demand is hedonic, in the spirit of Lancaster (1966): the household's utility is additively separable in the contributions of various *characteristics*, and a product provides a bundle of these characteristics. The Slutsky matrix $\boldsymbol{D}$ derived from this demand model can be expressed as a transformation of the cosine similarity matrix of products' characteristics, which Hoberg and Phillips (2016) estimated for a large set of consumer goods using text data.[29] We have calculated that in the Slutsky matrix derived this way, the eigenvalues are all small and the recoverable structure condition fails.[30] It is useful to reflect on why this is the case.

In the model of Pellegrino (2021), a simple calculation shows that it is impossible for the Slutsky matrix to have large eigenvalues.[31] For an economic intuition, note that in the Lancaster (1966) type of model, the "direct" relationship between any pair of goods is substitution. With substitution, if some demand is

[29]Pellegrino (2021) and Ederer and Pellegrino (2021) consider quantity competition, but the Slutsky matrix does not depend on this choice.

[30]There are more than 3000 products in the data and for the case of i.i.d. noise we would require that the largest eigenvalue is considerably larger than $b(n) = \sqrt{n} \approx 54$; this fails as largest eigenvalue of $\boldsymbol{D}$ in absolute value is about 2.

[31]The (un-normalized) Slutsky matrix in Pellegrino (2021) is $-\boldsymbol{B}^{-1}$, where $\boldsymbol{B} = \boldsymbol{I} + \alpha(\boldsymbol{\Sigma} - \boldsymbol{I})$. The matrix $\boldsymbol{\Sigma}$ is positive semidefinite because it can be written as $\boldsymbol{V}^{\mathsf{T}}\boldsymbol{V}$, where the columns of $\boldsymbol{V}$ are the characteristic vectors of various products. Thus all eigenvalues of $\boldsymbol{B}$ are real numbers bounded below by $1 - \alpha$, and all eigenvalues of $-\boldsymbol{B}^{-1}$ are at most $1/(1-\alpha)$ in magnitude. Pellegrino uses the value $\alpha = 0.12$, which prevents any eigenvalue from exceeding 1.13. We do not work with exactly the same Slutsky matrix because of the normalization in Appendix A.1. Its eigenvalues are a bit different, but they can still be bounded by a constant by elaborating this argument. Numerically we see that the normalization makes little difference.

diverted from one good due to an increase in its price, the total effect on all substitute goods is bounded, since, loosely speaking, the demand gained by these other goods must come out of the demand lost by the more expensive one. This bounds the sum of positive entries in $\boldsymbol{D}$ corresponding to this effect, which in turn bounds any complementarities in the Slutsky matrix.[32] In essence, in a hedonic model where the basic force is substitution, overall spillovers remain bounded, and the fact that $\boldsymbol{D}$ has no large eigenvalues is the mathematical manifestation of this.

Quite different behavior emerges in models where utility arises directly from consuming goods together, and such complementarities are central to our examples of recoverable structure. A leading practical example comes from the use of computers: a consumer's utility from a computer depends on the hardware, operating system, and applications. Two firms selling distinct components—a hardware device and an operating system, for instance—supply complementary goods, while two firms selling the same component (say, operating systems) supply substitute goods (Matutes and Regibeau, 1988, 1992). Our illustrative Example 2 in Section 5 shows how Slutsky matrices with large eigenvalues arise naturally in such settings.[33] But, as we have seen, it is impossible to produce the same patterns in models of the Lancaster (1966) type, because they cannot generate large eigenvalues; one would need to incorporate terms reflecting that some characteristics provide greater value when enjoyed together. There is a straightforward economic intuition for why such complementarities more readily produce recoverable structure: when one good's price decreases, all its complements can experience comparable nonvanishing increases in demand. This creates the clusters of nonvanishing entries in $\boldsymbol{D}$ that are the hallmark of recoverable structure.

In summary, direct complementarities seem practically important and can naturally yield the recoverable structure central to our results. We hope these observations will motivate further empirical research on the structure of large-scale oligopoly models with complementary goods.

7.3. **Games on networks.** One can view our exercise as a special case of an intervention, under noisy information, in a game among a large number of agents. In our case, the game comes from a standard oligopoly pricing model. Under the assumption of linear demand, the pricing game can be seen as a network game with linear best replies where the Slutsky matrix defines the network. Our analysis shows that if the oligopoly exhibits recoverable structure, then there are robust interventions for particular economic objectives. The methods we have developed can be extended to other settings. For example, in a public goods setting, interventions would aim to realign private marginal returns with social marginal returns. The literature has developed tools to understand how to do this when the authority has precise information on the

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[32] Note that complementarity (where one good's demand decreases in the price of the other) *can* arise in Pellegrino (2021) model. This happens through indirect effects: the substitute of my substitute can be my complement. However, since the "direct" substitution effect is bounded in magnitude, so are the indirect consequences.

[33] The complementarities there happen to be within-category, but that is not important for our point here.

spillovers causing the underprovision of public goods.[34] However, we know little about designing interventions under noisy information about such externalities. Similarly, in contracting for teams under moral hazard, network methods have recently been developed for locally perturbing contracts to achieve better outcomes for a principal (Dasaratha et al., 2024). But it is a considerable challenge to extend these results to the realistic case where the strategic interactions among members of an organization are only imperfectly known. General games will lack some of the structure we have leveraged, including the properties of the spillovers structure coming from a symmetric, positive semidefinite Slutsky matrix. So there are challenges to overcome in extending our results. We hope this paper stimulates research in these directions.

7.4. **Nonlinear demand.** We have assumed that demand is exactly linear in a neighborhood around the status quo equilibrium point. This assumption implies that the pass-through of an intervention to prices and quantities and, therefore, to welfare, depends only on the Slutsky matrix $\boldsymbol{D}$. We use this simplification to develop new concepts useful for robust market interventions. These concepts can be extended to nonlinear demand settings. We briefly explain how.

When demand is not locally linear, the pass-through of marginal cost shocks depends not only on the Slutsky matrix (which is the Jacobian of demand) but also on the Hessian of the demand function, the matrix whose (i, j) entry is $\partial^2 q_i(\boldsymbol{p})/\partial p_i \partial p_j$ (see, e.g., Miklos-Thal and Shaffer (2021)). Thus, our calculations would change, and there is no guarantee that our linear tax/subsidy interventions (based only on the Slutsky matrix at the status quo) would perform as they do in the linear model.

However, our main result can be extended once we allow the authority to use nonlinear interventions, i.e., to commit a vector of functions specifying a payment to each producer i as a function of all prices and quantities realized after the intervention. With this broader set of instruments, the authority can use nonlinear rebates based on post-intervention quantities to reduce the problem to the one we have studied. The key idea is to effectively linearize the demand the firms face around the status quo by using transfers to make up the difference between realized demand and a linear demand function. Once demand has been "linearized" in this way, the problem that firms face becomes equivalent to the one we have studied and we can use the results developed to design per-unit tax/subsidy interventions with desirable welfare properties. If we assume that the curvature of the demand of each product is locally bounded by a known constant, the payments needed to linearize demand can be bounded by a small fraction of the first-order gains of an intervention, so our welfare guarantees remain valid. Such assumptions on curvature also allow us to specify concrete sizes of interventions that achieve a given level of welfare gain, rather than just characterizing the behavior of derivatives.

[34] See, for instance, Bramoullé et al. (2014).

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APPENDIX A. OMITTED PROOFS AND DETAILS FOR MAIN RESULTS

A.1. **Normalization of spillover matrix.** For any differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, let $\mathcal{D}f(\boldsymbol{x})$ be the Jacobian matrix of the function evaluated at $\boldsymbol{x} \in \mathbb{R}^n$, whose (i,j) entry is $\partial f_i/\partial x_j$, where f_i denotes coordinate i of the function.

Here we will be explicit about distinguishing quantity variables $\boldsymbol{q}$ from the corresponding demand function; to this end, we will write the function as $\mathfrak{q}$.

Consider the change of coordinates for quantities given by $\widetilde{q}_i = \gamma_i q_i$. Keeping units of money fixed, the corresponding prices are $\widetilde{p}_i = p_i/\gamma_i$. Let $\boldsymbol{\Gamma}$ be the diagonal matrix whose (i,i) entry is γ_i. With these new units, we can define a function

$$\widetilde{\mathfrak{q}}(\widetilde{\boldsymbol{p}}) = \boldsymbol{\Gamma}\mathfrak{q}(\boldsymbol{\Gamma}\widetilde{\boldsymbol{p}})$$

and by the chain rule we have that

$$\mathcal{D}\widetilde{\mathfrak{q}}(\widetilde{\boldsymbol{p}}) = \boldsymbol{\Gamma}\left[\mathcal{D}\mathfrak{q}(\widetilde{\boldsymbol{p}})\right]\boldsymbol{\Gamma}.$$

For a given demand function $\mathfrak{q} : \mathbb{R}^N \to \mathbb{R}^N$, recall $\boldsymbol{D}$ is defined to be $\mathcal{D}\mathfrak{q}(\boldsymbol{p}^*)$, where $\boldsymbol{p}^*$ are equilibrium prices, uniquely determined under our maintained assumptions. We write $\boldsymbol{D}^{\mathfrak{q}}$ for $\mathcal{D}\mathfrak{q}(\boldsymbol{p}^*)$, It follows from this and the above paragraph that

$$\boldsymbol{D}^{\widetilde{\mathfrak{q}}} = \boldsymbol{\Gamma}\boldsymbol{D}^{\mathfrak{q}}\boldsymbol{\Gamma}.$$

Now set $\gamma_i = 1/\sqrt{|D^{\mathfrak{q}}_{ii}|}$. It is clear from the above formula that $\boldsymbol{D}^{\widetilde{\mathfrak{q}}}$ has -1 on the diagonal.

Thus, under a suitable choice of units, the matrix $\boldsymbol{D}$ may be assumed to have diagonal -1.

A.2. **Proof of Fact 1.** Fix an n and a set of types $\{1, 2, \ldots, m\}$, with the number of products of type t being $N(t)$.

The matrix $\boldsymbol{D}$ can be written as

$$\boldsymbol{D} = \boldsymbol{A} + (1 + \gamma(n))\boldsymbol{I}$$

where $\boldsymbol{A}$ is a block matrix, in the sense that A_{ij} depends only on $k(i)$ and $k(j)$.

The nonzero eigenvalues of $\boldsymbol{A}$ are the same as those of the m-by-m matrix $\widetilde{\boldsymbol{A}}$ whose (t,t') entry is

$$\widetilde{A}_{tt'} = \gamma(n)\sqrt{N(t)N(t')}\bar{D}_{tt'}.$$

There is some t so that $N(t) \geq n/|M|$. Thus,

$$|\widetilde{A}_{tt'}| \geq \frac{\gamma(n)}{|M|} n |\bar{D}_{tt}| = \frac{\gamma(n)}{|M|} n,$$

recalling $\bar{\boldsymbol{D}}$ has diagonal -1. By the Courant–Fisher Theorem, this is a lower bound on the magnitude of the largest eigenvalue $\widetilde{\boldsymbol{A}}$. Since the adjustment coming from the scaled identity matrix is of lower order, $\boldsymbol{D}$ has an eigenvalue at least $\gamma(n)n/2|M|$ in magnitude.

Now let $\widetilde{\boldsymbol{u}}^1$ be the corresponding eigenvector of $\widetilde{\boldsymbol{A}}$ normalized to lie in the unit ball, and $\widetilde{\boldsymbol{v}}^1$ be its limit as $n \to \infty$. This limit depends only on the limit $\widetilde{\boldsymbol{A}}/n$, which exists by our assumption that $N(t)/n$ converges for each n. The limit projection of $\boldsymbol{q}^0$ onto $\boldsymbol{u}^1$ depends only on $\bar{\boldsymbol{q}}$ and $\boldsymbol{v}^1$, and for generic $\boldsymbol{q}$ this is nonzero.

A.3. **Proof of Theorem 1.** We prove the theorem under Assumptions 1–3 but we replace Assumption 4 with the following weaker assumption:

Assumption 5. We assume that:
(1) $\mathbb{E}[\|\boldsymbol{E}\|] \leq b(n)$;
(2) for any sequence of linear subspaces $V(n)$ of $\mathbb{R}^n$ with dimension $d(n)$, where $d(n)/n \to_n 0$, the norm $\|P_{V(n)}\boldsymbol{\varepsilon}\|$ tends to 0 in probability.

Note that parts (1) of Assumption 4 and Assumption 5 are identical. However, part (2) of Assumption 4, which stated that all ε_i are independent, implies part (2) of Assumption 5.

Some notation: For any matrix $\boldsymbol{M}$, we define $\mathbb{A}(\boldsymbol{M}, \underline{\lambda})$ as the set of eigenvalues of $\boldsymbol{M}$ with absolute value greater than or equal to $\underline{\lambda}$ and $\mathcal{L}(\boldsymbol{M}, \underline{\lambda})$ as the space spanned by corresponding eigenvectors.

Recall that under Assumption 5, a signal with $b(n)$-bounded noise can be written $\widehat{\boldsymbol{D}} = \boldsymbol{D}(\boldsymbol{p}^0) + \boldsymbol{E}$, where, by Markov's inequality,

$$\|\boldsymbol{E}\| \leq kb(n) \tag{14}$$

with high probability, where k is a large constant; we can absorb this constant into $b(n)$, which we will do from now on.

Recall also the conditions of the theorem imply that we have sequences $b(n)$ and $M(n)$ such that, for large enough n, $\boldsymbol{D}$ has eigenvalues exceeding $M(n)$ where[35] $b(n) \ll M(n)$ and $\mathbb{A}(\boldsymbol{D}, M(n))$ is nonempty. We also choose two other sequences $\underline{M}(n)$ and $\widehat{M}(n)$, such that $\underline{M}(n) \ll \widehat{M}(n) \ll a(n)$ and the differences between these successive sequences also dominate $b(n)$.

We use the following notation:

$$\underline{\Lambda}(n) := \mathbb{A}(\boldsymbol{D}; \underline{M}(n)), \quad \widehat{\Lambda}(n) := \mathbb{A}(\widehat{\boldsymbol{D}}; \widehat{M}(n)), \quad \Lambda(n) := \mathbb{A}(\boldsymbol{D}; M(n))$$
$$\underline{L}(n) := \mathcal{L}(\boldsymbol{D}, \underline{M}(n)), \quad \widehat{L}(n) := \mathcal{L}(\widehat{\boldsymbol{D}}, \widehat{M}(n)), \quad L(n) := \mathcal{L}(\boldsymbol{D}, M(n)).$$

Let P_V be the projection operator onto subspace V and $P_V^{\perp}$ its orthogonal complement. Let $(\lambda_1, \boldsymbol{u}^1), \ldots, (\lambda_n, \boldsymbol{u}^n)$ be eigenpairs of $\boldsymbol{D}$, with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

We now define our intervention:

$$\boldsymbol{\sigma} = \frac{P_{\widehat{L}(n)}\widehat{\boldsymbol{q}}^0}{\|P_{\widehat{L}(n)}\widehat{\boldsymbol{q}}^0\|^2} \tag{15}$$

The expenditure of this intervention is

$$\dot{S} = \boldsymbol{\sigma} \cdot \boldsymbol{q}^0 = \frac{P_{\widehat{L}(n)}\widehat{\boldsymbol{q}}^0}{\|P_{\widehat{L}(n)}\widehat{\boldsymbol{q}}^0\|^2} \cdot \boldsymbol{q}^0 \tag{16}$$

Our first main lemma, which we will prove shortly, will assert that $\dot{S}$ converges in probability to 1. The challenge in proving such a result is that the actual expenditure depends on true quantities, whereas the intervention is built based on *estimated* quantities projected onto an "estimated" eigenspace of $\widehat{\boldsymbol{D}}$. We begin with a technical result that will be key to controlling the differences between actual and estimated objects. It relies on the Davis–Kahan Theorem.

Lemma 4. The norm $\|P^{\perp}_{\underline{L}(n)} P_{\widehat{L}(n)}\|$ converges to 0 in probability as $n \to \infty$.

[35]We use the notation $a(n) \ll b(n)$ to mean that $a(n)/b(n) \to_n 0$.

Proof. The Davis–Kahan Theorem guarantees that, for any $\eta > 0$, there exists $N_1(\eta)$ such that for all $n > N_1(\eta)$, with probability greater than or equal to $1 - \eta$, two properties hold. First, every eigenvalue in the set $\Lambda(n)$, which is nonempty by the recoverable structure assumption, has a corresponding eigenvalue within distance $O(b(n))$, and therefore in $\widehat{\Lambda}(n)$. Second,

$$\|P^{\perp}_{\underline{L}(n)} P_{\widehat{L}(n)}\| \leq \frac{2\|\boldsymbol{E}\|}{g}, \tag{17}$$

where g, the "gap," is the minimum distance between some eigenvalue of $\widehat{\boldsymbol{D}}$ in $\widehat{\Lambda}(n)$ and some eigenvalue of $\boldsymbol{D}$ not contained in $\underline{\Lambda}(n)$. This gap is at least $\widehat{M}(n) - \underline{M}(n) \gg b(n)$. Thus, increasing $N_1(\eta)$ if necessary, we conclude the statement of the lemma. □

Lemma 5. As $n \to \infty$, the expenditure derivative $\dot{S}$ converges in probability to 1.

Proof. Write $\boldsymbol{q}^0 = \widehat{\boldsymbol{q}}^0 - \boldsymbol{\varepsilon}$ and calculate

$$\begin{aligned}\dot{S} = \boldsymbol{\sigma} \cdot \boldsymbol{q}^0 &= \frac{P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0}{\|P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0\|^2} \cdot (\widehat{\boldsymbol{q}}^0 - \boldsymbol{\varepsilon}) \\ &= 1 - \frac{P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0}{\|P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0\|^2} \cdot P_{\widehat{L}(n)} \boldsymbol{\varepsilon}\end{aligned}$$

By the Cauchy–Schwarz inequality

$$|P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0 \cdot P_{\widehat{L}(n)} \boldsymbol{\varepsilon}| \leq \|P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0\| \cdot \|P_{\widehat{L}(n)} \boldsymbol{\varepsilon}\|,$$

which, combined with the above, gives

$$\dot{S} \geq 1 - \frac{\|P_{\widehat{L}(n)} \boldsymbol{\varepsilon}\|}{\|P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0\|}.$$

The fact that $L(n)$ is a subspace of $\widehat{L}(n)$ and the assumption of recoverable structure together ensure that $\|P_{\widehat{L}(n)} \widehat{\boldsymbol{q}}^0\| \geq \delta$.

So to obtain the conclusion, it suffices to show that $\|P_{\widehat{L}(n)} \boldsymbol{\varepsilon}\|$ tends in probability to 0. This will be established via Assumption 5. To apply it, we need to show that $\dim \widehat{L}(n)/n \to_n 0$. The reason this holds is that if there were more than $n/M(n)$ eigenvalues in $\Lambda(\boldsymbol{D}, M(n))$, then the absolute value of sum of these eigenvalues would exceed n, but the trace (and hence sum of eigenvalues) of the negative semidefinite matrix $\boldsymbol{D}$ is $-n$. Since $M(n) \to \infty$, the statement is established. □

Now, to prove the theorem, we use Lemma 3 to write:

$$\dot{W} = \underbrace{\sum_{\lambda_\ell \in \underline{\Lambda}(n)} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) \frac{|\lambda_\ell|}{1 + |\lambda_\ell|}}_{\dot{W}_M} + \underbrace{\sum_{\lambda_\ell \notin \underline{\Lambda}(n)} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) \frac{|\lambda_\ell|}{1 + |\lambda_\ell|}}_{\dot{W}_R}. \tag{18}$$

where we have divided the quantity into a main (M) part and the rest (R). A similar (but simpler) decomposition applies to expenditure

$$\dot{S} = \underbrace{\sum_{\lambda_\ell \in \underline{\Lambda}(n)} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma})}_{\dot{S}_M} + \underbrace{\sum_{\lambda_\ell \notin \underline{\Lambda}(n)} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma})}_{\dot{S}_R}. \tag{19}$$

The proof can be completed with two further lemmas.

Lemma 6. As $n \to \infty$, both $\dot{W}_R$ and $\dot{S}_R$ converge in probability to 0.

Proof. This follows from Lemma 4 and the fact that by construction, $\boldsymbol{\sigma} \in \widehat{L}(n)$. □

Lemma 7. As $n \to \infty$,

$$\dot{W}_M - \dot{S}_M \xrightarrow{p} 0.$$

Proof. Using the expressions above, write the difference

$$|\dot{W} - \dot{S}| = \sum_{\lambda_\ell \in \underline{\Lambda}(n)} \left[\frac{|\lambda_\ell|}{1 + |\lambda_\ell|} - 1 \right] (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}).$$

Note that for all $\ell \in \underline{\Lambda}(n)$, $|\lambda_\ell| \geq \underline{M}(n)$, so $1 - \frac{1}{\underline{M}(n)} \leq \frac{|\lambda_\ell|}{1+|\lambda_\ell|} \leq 1$. By Hölder's inequality (with $p = \infty$ and $q = 1$), we have

$$|\dot{W} - \dot{S}| \leq \frac{1}{\underline{M}(n)} \sum_{\lambda_\ell \in \underline{\Lambda}(n)} |(\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma})|.$$

Then applying Cauchy-Schwarz term by term, we have

$$|\dot{W} - \dot{S}| \leq \frac{1}{\underline{M}(n)} \sum_{\lambda_\ell \in \underline{\Lambda}(n)} \|P_{\boldsymbol{u}^\ell} \boldsymbol{q}^0\| \|P_{\boldsymbol{u}^\ell} \boldsymbol{\sigma}\|.$$

$$|\dot{W} - \dot{S}| \leq \frac{1}{\underline{M}(n)} \|P_{\underline{L}(n)} \boldsymbol{q}^0\| \|P_{\underline{L}(n)} \boldsymbol{\sigma}\|.$$

Now using our Assumption 3—that $\|\boldsymbol{q}^0\|$ is bounded in n—gives the result. □

We can put everything together. Lemma 5 gives that $\dot{S} \xrightarrow{p} 1$. Combining this with (19) and Lemma 6 (which says that $\dot{S}_R \xrightarrow{p} 0$) we see that $\dot{S}_M \xrightarrow{p} 1$. Then using Lemma 7, we find that $\dot{W}_M \xrightarrow{p} 1$. Another application of Lemma 6 gives that $\dot{W}_R \xrightarrow{p} 0$, so that $\dot{W} \xrightarrow{p} 1$. The claim about the effect on $\dot{C}$ follows immediately from Proposition 1.

Finally, we consider the effect on individual consumer surpluses. For any consumer h,

$$\dot{C}^h = -\boldsymbol{q}^h \cdot \dot{\boldsymbol{p}} = -\sum_{\ell=1}^{n} (\boldsymbol{u}^\ell \cdot \boldsymbol{q}^h)(\boldsymbol{u}^\ell \cdot \dot{\boldsymbol{p}}) \tag{20}$$

Using Lemma 2:

$$\boldsymbol{u}^\ell \cdot \dot{\boldsymbol{p}} = \frac{1}{1 + |\lambda_\ell|} (\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) \tag{21}$$

Then by an argument very similar to the proof of Lemma 6, we conclude that $\dot{C}^h \xrightarrow{p} 0$.

A.4. **Proof of Proposition 3.** We prove each part separately.

Part 1: We construct an environment (i.e., a set of market states $\boldsymbol{\Theta}$ and, for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ a distribution $\mu_{\boldsymbol{\theta}}$ of $(\widehat{\boldsymbol{D}}, \hat{\boldsymbol{q}}^0)$) where no intervention robustly increases total surplus, but where an omniscient authority could achieve total surplus gains $\dot{P} + \dot{C} = 2\dot{S}$ with $\dot{C} \geq 0$.

Let

$$\boldsymbol{D}^* = -\frac{n}{n-1}\boldsymbol{I} + \frac{1}{n-1}\boldsymbol{1}\boldsymbol{1}^\top$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix and $\boldsymbol{1}$ is the n-dimensional vector of ones. Let $Q \subset \mathbb{R}^n$ be the $(n-1)$-dimensional sphere of vectors orthogonal to $\boldsymbol{1}$ with norm 1. The set of market states is

$$\boldsymbol{\Theta} = \left\{ (\boldsymbol{D}, \boldsymbol{q}^0) \,:\, \boldsymbol{D} = \boldsymbol{D}^* \text{ and } \boldsymbol{q}^0 = \frac{1}{n}\left(\boldsymbol{1} + \frac{1}{2}\boldsymbol{r}\right), \text{ for some } \boldsymbol{r} \in Q. \right\}$$

In this proof, it will be useful to define random variables on the extended reals $\mathbb{R} \cup \{-\infty, \infty\}$, which allows us to take $\varepsilon_i = \infty$ for all i.[36] Note that $\boldsymbol{D}$ is known and the signal about $\boldsymbol{q}^0$ is always ∞ in every entry, so that any intervention rule implements a single intervention—call it $\boldsymbol{\sigma}^*$.

Note that $\boldsymbol{D}$ is a normalized Slutsky matrix. The spectral decomposition is as follows:

- $\lambda_1 = 0$ with multiplicity 1; the corresponding eigenvector is $\boldsymbol{u}^1 = n^{-1/2}\boldsymbol{1}$;
- $\lambda_2 = -\frac{n}{n-1}$ with multiplicity $n-1$.

Now write

$$\boldsymbol{\sigma} = a_1 \boldsymbol{u}^1 + a_2 \boldsymbol{u}^2, \tag{22}$$

where $\boldsymbol{u}^2$ is a vector orthogonal to $\boldsymbol{u}^1$ and $a_2 \leq 0$ (we can achieve this sign of a_2 by choosing the sign of the vector $\boldsymbol{u}^1$ appropriately).

Complete $(\boldsymbol{u}^1, \boldsymbol{u}^2)$ to an orthonormal basis of eigenvectors of $\boldsymbol{D}$, $(\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}^3, \ldots, \boldsymbol{u}^n)$. Consider the market state $(\boldsymbol{D}, \boldsymbol{q}^0) \in \boldsymbol{\Theta}$ given by $\boldsymbol{D} = \boldsymbol{D}^*$ and

$$\boldsymbol{q}^0 = \frac{1}{n}\left(\boldsymbol{1} + \frac{1}{2}\boldsymbol{u}^2\right).$$

For any intervention $\boldsymbol{\sigma}$, Lemma 3 states that the change in total surplus is given by:

$$\dot{W} = -\sum_{\ell=1}^{n} \frac{|\lambda_\ell|}{1 + |\lambda_\ell|}(\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) \tag{23}$$

It follows by our choice of $\boldsymbol{q}^0$ that $\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}$ is equal to a_1 and a_2 for $\ell = 1$ and $\ell = 2$ respectively, and to 0 for $\ell > 2$. Also, $\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0$ is equal to $n^{-1/2}$ for $\ell = 1$ and to $n^{-1}/2$ for $\ell = 2$. Since $\lambda_1 = 0$, only the term corresponding to $\ell = 2$ is nonzero, and so

$$\dot{W} = -\frac{|\lambda_2|}{1 + |\lambda_2|}\frac{1}{2n}a_2,$$

which is non-positive by our choice of $a_2 \leq 0$. Therefore, in this market state $\boldsymbol{\sigma}$ achieves $\dot{W} > 0$ with probability 0. We have thus shown that no intervention can achieve $\dot{W} > 0$ ϵ–robustly, for any $\epsilon > 0$.

[36]In Assumption 4, we may take $\overline{V} = \infty$; this is immaterial since that assumption plays a role only in our positive results. Since $\boldsymbol{D}$ is deterministic, we can take $\boldsymbol{E}$ to be the zero matrix.

On the other hand, by Proposition 4 in Appendix B, any surplus outcome $(\dot{P}, \dot{C}, \dot{S})$ satisfying $\frac{1}{2}\dot{P} + \dot{C} = \dot{S}$ can be achieved.

Part 2: Note the setting here is different from Part 1: in that case, we were constructing an environment without aggregate structure, while here we will give an example of an environment with recoverable structure where no intervention can robustly increase consumer surplus.

The set of market states is as follows. Since we have to show that there is no intervention that has the claimed property for all n, we will work with $n = 2^m$ firms, where m is a positive integer. Let $\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^n \in \{-1, +1\}^n$ be vectors that are mutually orthogonal, with $\boldsymbol{r}^1$ being the vector of all ones. (This is possible by Sylvester's construction of Hadamard matrices.) Let $\boldsymbol{u}^\ell = \frac{1}{\sqrt{n}}\boldsymbol{r}^\ell$. Let

$$\boldsymbol{D}^* = -\frac{n}{2}(\boldsymbol{u}^1)^\top \boldsymbol{u}^1 - \frac{1}{2}\boldsymbol{I}.$$

Note this matrix has eigenvalues $\lambda_1 = -\frac{n+1}{2}$ with multiplicity 1 and $-1/2$ with multiplicity $n-1$. The set of possible market states is $(\boldsymbol{D}^*, \boldsymbol{q}^0)$, where $\boldsymbol{q}^0$ is given by any vector of the form

$$\boldsymbol{q}^0 = \frac{1}{2}\left(\boldsymbol{u}^1 + f(n)\sum_{\ell=2}^{n} s_\ell \boldsymbol{u}^\ell\right), \tag{24}$$

where each $s_\ell \in \{-1, +1\}$, and $f(n)$ is a real-valued function we will specify. Each $(\boldsymbol{D}^*, \boldsymbol{q}^0) \in \boldsymbol{\Theta}$ satisfies our maintained assumptions. Moreover, $\boldsymbol{\Theta}$ has $(M(n), \delta)$-recoverable structure for any $M(n) \leq n/2$ and $\delta = .49$.

In any environment, suppose there is an intervention rule that achieves the property $\dot{C} \geq \varepsilon > 0$ robustly over a set of market states $\boldsymbol{\Theta}$. Then, by definition of robustly achieving the property, the intervention rule must achieve it with high probability when $\boldsymbol{\theta}$ is drawn from any distribution over $\boldsymbol{\Theta}$. We will use this fact now, specifying the distribution over $\boldsymbol{\Theta}$ given by drawing the s_ℓ in (24) with equal probability from ± 1, and derive a contradiction to our assumption at the start of this paragraph.

Recall

$$\dot{C} = -\sum_{\ell=1}^{n} \frac{1}{1 + |\lambda_\ell|}(\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}).$$

The $\ell = 1$ converges in probability to 0 as $n \to \infty$, since $|\lambda_1| \to \infty$. Note we may take each $\boldsymbol{u}^\ell$ to be a scaling of $\boldsymbol{r}^{(\ell)}$.

To handle the rest of the summation, we introduce the error structure in this environment: let the ε_i be independent and normal, each with variance $1/n$, which satisfies Assumption 4. We note a useful fact. Fixing any $\delta > 0$ and n, if $f(n)$ is small enough, then (under Assumption 4), for large enough n, the conditional distribution over $\boldsymbol{q}^0 - \frac{1}{2}\boldsymbol{u}^1$ given the authority's signal (in this example just $\widehat{\boldsymbol{q}}^0$) is within δ of the prior in total variation norm, except for signals having probability at most δ.[37]

[37] Let's work in the basis $(\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^n)$. For each ℓ, the authority observes a signal $z_\ell = f(n)s_\ell \boldsymbol{u}^\ell + \zeta_\ell$, where ζ_ℓ are i.i.d. normal with variance $1/n$ (we have used that the rotation into the new basis does not affect the distribution of errors). Now fix n and condition on error realizations satisfying $\|\boldsymbol{\varepsilon}\| \leq K(n)$, where $K(n)$ will be specified later. For a fixed $K(n)$, if $f(n)$

Now, let

$$\dot{C}_{\geq 2} = -\sum_{\ell=2}^{n} \frac{1}{1+|\lambda_\ell|}(\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) = -\frac{2}{3}\sum_{\ell=2}^{n}(\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0)(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}).$$

We will study the distribution of this random variable for any fixed $\boldsymbol{\sigma}$, under the prior. We claim that $\dot{C}_{\geq 2}$ has the same distribution as $-\dot{C}_{\geq 2}$. This is because in the above expression, only $\boldsymbol{q}^0$ is random, and $\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0$ are independently equal to plus or minus the same constant by definition. Thus, for any $\epsilon > 0$,

$$P(\dot{C}_{\geq 2} \leq -\epsilon \mid \widehat{\boldsymbol{q}}^0) - P(\dot{C}_{\geq 2} \geq \epsilon \mid \widehat{\boldsymbol{q}}^0) \to_n 0.$$

Now, taking expectations over $\widehat{\boldsymbol{q}}^0$, and using the useful fact, we find that

$$P(\dot{C}_{\geq 2} \leq -\epsilon) - P(\dot{C}_{\geq 2} \geq \epsilon) \to_n 0.$$

Thus we have a contradiction to robustly achieving $\dot{C} \geq \epsilon$.

On the other hand, again by Proposition 4 in Appendix B, any surplus outcome $(\dot{P}, \dot{C}, \dot{S})$ satisfying $\frac{1}{2}\dot{P} + \dot{C} = \dot{S}$ can be achieved.

Remark 3. For simplicity, this proof has worked with a case where all uncertainty comes from $\boldsymbol{q}^0$, and the projections of $\boldsymbol{q}^0$ onto various low eigenvectors are small. A stronger negative result can be obtained by elaborating the construction a bit—making the λ_ℓ different from one another, so that the eigenspaces orthogonal to $\boldsymbol{1}$ are not deterministic. Then, by showing that the eigenvectors are often impossible to recover with any precision, we can obtain a similar conclusion even if all the projections $\boldsymbol{u}^\ell \cdot \boldsymbol{q}^0$ are bounded below.

Appendix B. Complete information

Recall from Proposition 1 that

$$\frac{1}{2}\dot{P} + \dot{C} = \dot{S} \tag{25}$$

always holds. We now show the following result.

Proposition 4. If all products are independent (i.e., $\boldsymbol{D} = -\boldsymbol{I}$), then an intervention that spends $\dot{S}$ dollars implements $\dot{C} = \frac{\dot{S}}{2}$ and $\dot{P} = \dot{S}$. Otherwise, for generic $\boldsymbol{q}^0$, any $(\dot{C}, \dot{P}, \dot{S})$ that satisfies (25) can be implemented by an intervention.

Proof. First, consider the case in which products are independent. In this case, the effect of a subsidy to firm i and a subsidy to firm j to surpluses are separable. So it is sufficient to prove the result for an intervention which only subsidizes one firm: take $\boldsymbol{\sigma}$ with $\sigma_j = 0$ for all $j \neq 1$ and $\sigma_1 > 0$, so that $\dot{S} = \sigma_1 q_1^0$. Note the basis diagonalizing $\boldsymbol{D}$ is the standard basis, and using the formulas in Lemma 3 with $\lambda_\ell = -1$ for all ℓ, we conclude $\dot{C} = \sigma q_1^0/2$ and $\dot{P} = \sigma q_1^0$, as claimed.

Second, if products are not all independent ($\boldsymbol{D} \neq -\boldsymbol{I}$), then there exists $\ell \neq \ell'$ with $\lambda_\ell \neq \lambda'_\ell$. For generic $\boldsymbol{q}$, we also have that $(\boldsymbol{u}^\ell \cdot \boldsymbol{q})(\boldsymbol{u}^\ell \cdot \boldsymbol{q}') \neq 0$. Without loss, take $\ell = 1$ and $\ell' = 2$. We now construct a set of interventions under which we can obtain any outcome $(\dot{C}, \dot{P}, \dot{S})$ that satisfies (25).

is chosen small enough, the density of the signal z_ℓ depends arbitrarily little on s_ℓ, and thus the posterior distribution about s_ℓ depends arbitrarily little on the signal.

For any real number $\beta \neq 0$ and any $\dot{S}$, consider the intervention

$$\boldsymbol{\sigma} = \beta\left((\boldsymbol{u}^1 \cdot \boldsymbol{q})\boldsymbol{u}^1 + \alpha(\boldsymbol{u}^2 \cdot \boldsymbol{q})\boldsymbol{u}^2\right),$$

where α is chosen so that $\boldsymbol{\sigma} \cdot \boldsymbol{q} = \dot{S}$. Note that $\boldsymbol{\sigma} \cdot \boldsymbol{q} = \beta((\boldsymbol{u}^1 \cdot \boldsymbol{q})^2 + \alpha(\boldsymbol{u}^2 \cdot \boldsymbol{q})^2)$ so this implies that

$$\alpha = \frac{\dot{S}/\beta - (\boldsymbol{u}^1 \cdot \boldsymbol{q})^2}{(\boldsymbol{u}^2 \cdot \boldsymbol{q})^2}$$

Hence, as we vary β, we keep $\boldsymbol{\sigma} \cdot \boldsymbol{q}$ constant and equal to $\dot{S}$. Next, note that

$$\begin{aligned}
\dot{C} &= \sum_{\ell} \frac{1}{1+|\lambda_\ell|}(\boldsymbol{u}^\ell \cdot \boldsymbol{q})(\boldsymbol{u}^\ell \cdot \boldsymbol{\sigma}) \\
&= \frac{\beta}{1+|\lambda_1|}(\boldsymbol{u}^1 \cdot \boldsymbol{q})^2 + \frac{\alpha\beta}{1+|\lambda_2|}(\boldsymbol{u}^2 \cdot \boldsymbol{q})^2 \\
&= \frac{\beta}{1+|\lambda_1|}(\boldsymbol{u}^1 \cdot \boldsymbol{q})^2 + \frac{\dot{S} - \beta(\boldsymbol{u}^1 \cdot \boldsymbol{q})^2}{(\boldsymbol{u}^2 \cdot \boldsymbol{q})^2}\frac{1}{1+|\lambda_2|}(\boldsymbol{u}^2 \cdot \boldsymbol{q})^2 \\
&= \frac{\beta}{1+|\lambda_1|}(\boldsymbol{u}^1 \cdot \boldsymbol{q})^2 + \frac{\dot{S} - \beta(\boldsymbol{u}^1 \cdot \boldsymbol{q})^2}{1+|\lambda_2|} \\
&= \beta(\boldsymbol{u}^1 \cdot \boldsymbol{q})^2\left(\frac{1}{1+|\lambda_1|} - \frac{1}{1+|\lambda_2|}\right) + \frac{1}{1+|\lambda_2|}\dot{S}
\end{aligned}$$

Since this is a non-constant linear function in β, for a given $\dot{S}$, we can achieve any given $\dot{C}$ by choosing β appropriately. □

APPENDIX C. A NOISE STRUCTURE UNDER WHICH ASSUMPTION 4 HOLDS

For a concrete interpretation of Assumption 4, we will present a sampling procedure and an associated estimator for the normalized demand matrix $\boldsymbol{D}$. As in Appendix A.1, we will not assume any normalization to begin with (since the authority does not have the luxury of the market being normalized).

Since all our notation about the Slutsky matrix (D_{ij}, its estimators $\widehat{D}_{ij}$, etc.) was for the normalized version of this matrix, for this proof we use underlines to refer the un-normalized matrix, whose entries are $\underline{D}_{ij} = \partial q_i / \partial p_j$. We will also use underlines in our notation for estimators of the un-normalized entries.

For simplicity, we assume that all households share a single representative utility function for goods and that the number of households exceeds n^2, where n is the number of firms. Moreover, we assume that, for all i and j, the entries $\underline{D}_{ij}$ are uniformly bounded, i.e., $|\underline{D}_{ij}| \leq \underline{D}_{\max} < \infty$ for some positive constant $\underline{D}_{\max}$. Also assume the true diagonal entries $\underline{D}_{ii}$ are bounded away from 0: $0 < \underline{D}_{\min} \leq |\underline{D}_{ii}|$ for all i. The constants $\underline{D}_{\min}$ and $\underline{D}_{\max}$ do not depend on i, j, or n.

For each product pair (i, j), the authority samples a distinct household—call it $h(i, j)$—with the representative preferences; it performs a demand experiment to obtain an unbiased estimate $\widehat{\underline{D}}_{ij}^{h(i,j)}$ of $\underline{D}_{ij}$ and an unbiased estimate $\widehat{\underline{D}}_{ii}^{h(i,j)}$ of $\underline{D}_{ii}$. Note there is a distinct estimate of $\underline{D}_{ii}$ for each household $h(i, j)$.

We drop the superscript $h(i, j)$ on $\widehat{\underline{D}}_{ij}^{h(i,j)}$ and write

$$\widehat{\underline{D}}_{ij} = \underline{D}_{ij} + F_{ij}.$$

We assume that the F_{ij} are mean-zero and independent across (i, j), that their support is uniformly bounded, and that the matrix $\boldsymbol{F}$ is symmetric.[38] Let $\overline{V}$ be an upper bound on $\mathrm{Var}[F_{ij}]$ that does not depend on i, j, or n.

Turning to estimates of $\underline{D}_{ii}$, we write

$$\widehat{\underline{D}}_{ii}^{h(i,j)} = \underline{D}_{ii} + G_{ij}^{h(i,j)}.$$

We assume that the $G_{ij}^{h(i,j)}$ are mean-zero and independent across (i, j), and that $G_{ij}^{h(i,j)}$ is independent of $F_{i'j'}$ unless $(i', j') = (i, j)$. We also assume that the variances of the $G_{ij}^{h(i,j)}$ are bounded by some number $\overline{V}$ that does not depend on i, j, or n. Now, letting

$$\widehat{\underline{D}}_{ii} = \frac{1}{n} \sum_j \widehat{\underline{D}}_{ij}^{h(i,j)},$$

we may write

$$\widehat{\underline{D}}_{ii} = \underline{D}_{ii} + G_{ii}, \text{ where } G_{ii} = \frac{1}{n} \sum_j G_{ij}^{h(i,j)}.$$

Next, as in Appendix A.1, let $\widehat{\boldsymbol{\Gamma}}$ be the diagonal matrix whose (i, i) diagonal entry is $\widehat{\Gamma}_{ii} = 1/\sqrt{\widehat{\underline{D}}_{ii}}$, and construct

$$\widehat{\boldsymbol{D}} = \widehat{\boldsymbol{\Gamma}} \widehat{\underline{\boldsymbol{D}}} \widehat{\boldsymbol{\Gamma}}.$$

[38] Since $\boldsymbol{D}$ is a symmetric matrix, we view the parameters as being the numbers on or above the diagonal.

Next, turning to true quantities (as opposed to hatted estimators) recall the analogous definitions $\Gamma_{ii} = 1/\sqrt{D_{ii}}$ and also recall that

$$\boldsymbol{D} = \boldsymbol{\Gamma}\underline{\boldsymbol{D}}\boldsymbol{\Gamma}$$

is the true normalized Slutsky matrix. Write $\boldsymbol{E} = \widehat{\boldsymbol{D}} - \boldsymbol{D}$. Notice that $\mathrm{Var}[G_{ii}] \leq \overline{V}/(n-1)$, a fact we will use throughout. (The reason for the very precise estimate is that we are using $n-1$ different, independent households' signals to estimate the same number $\underline{D}_{ii}$.)

Lemma 8. Under the sampling procedure described, $\mathbb{E}[\|\boldsymbol{E}\|] < b(n)$, where $b(n) = \gamma n^{1/2}$ for some constant $\gamma > 0$.

Proof. Let $\boldsymbol{\Phi} = \widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}$. We start by rewriting the error matrix $\boldsymbol{E}$:

$$\boldsymbol{E} = \widehat{\boldsymbol{D}} - \boldsymbol{D} = \widehat{\boldsymbol{\Gamma}}\,\widehat{\underline{\boldsymbol{D}}}\,\widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Gamma}.$$

Substituting $\widehat{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} + \boldsymbol{\Phi}$ and $\widehat{\underline{\boldsymbol{D}}} = \underline{\boldsymbol{D}} + \boldsymbol{F}$, we have:

$$\boldsymbol{E} = (\boldsymbol{\Gamma} + \boldsymbol{\Phi})(\underline{\boldsymbol{D}} + \boldsymbol{F})(\boldsymbol{\Gamma} + \boldsymbol{\Phi}) - \boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Gamma}.$$

Expanding the product and using the symmetry of all the matrices involved, we obtain:

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Gamma} + \boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Phi} + \boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Gamma} + \boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Phi} \\ &\quad + \boldsymbol{\Phi}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Gamma} + \boldsymbol{\Phi}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Phi} + \boldsymbol{\Phi}\,\boldsymbol{F}\,\boldsymbol{\Gamma} + \boldsymbol{\Phi}\,\boldsymbol{F}\,\boldsymbol{\Phi} - \boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Gamma} \\ &= \boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Gamma} + 2\,\boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Phi} + 2\,\boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Phi} + \boldsymbol{\Phi}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Phi} + \boldsymbol{\Phi}\,\boldsymbol{F}\,\boldsymbol{\Phi}.\end{aligned}$$

Our goal is to bound the expected spectral norm of $\boldsymbol{E}$:

$$\mathbb{E}[\|\boldsymbol{E}\|] = O(\sqrt{n}).$$

Using the triangle inequality for the spectral norm $\|\cdot\|$, we have:

$$\|\boldsymbol{E}\| \leq \|\boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Gamma}\| + 2\|\boldsymbol{\Gamma}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Phi}\| + 2\|\boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Phi}\| + \|\boldsymbol{\Phi}\,\underline{\boldsymbol{D}}\,\boldsymbol{\Phi}\| + \|\boldsymbol{\Phi}\,\boldsymbol{F}\,\boldsymbol{\Phi}\|.$$

We will bound the expected value of each term on the right-hand side. In this argument, we adopt the standard practice that the meaning of the constant C can change from line to line, but this symbol always stands for a deterministic constant that does not depend on i, j, or n.

First term: $\mathbb{E}[\|\boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Gamma}\|]$**.** Since $\boldsymbol{\Gamma}$ is a diagonal matrix with entries $\Gamma_{ii} = 1/\sqrt{D_{ii}}$, and D_{ii} is bounded away from zero and infinity, there exist constants $\Gamma_{\min}, \Gamma_{\max} > 0$ such that:

$$\Gamma_{\min} \leq \Gamma_{ii} \leq \Gamma_{\max}.$$

Thus, $\|\boldsymbol{\Gamma}\| \leq \Gamma_{\max}$. The matrix $\boldsymbol{F}$ has independent, mean-zero entries F_{ij} with variances $\mathrm{Var}(F_{ij}) \leq \overline{V}$.

Consider the matrix $\boldsymbol{H} = \boldsymbol{\Gamma}\,\boldsymbol{F}\,\boldsymbol{\Gamma}$ with entries $H_{ij} = \Gamma_{ii}F_{ij}\Gamma_{jj}$. The variances of H_{ij} satisfy:

$$\mathrm{Var}(H_{ij}) = \Gamma_{ii}^2\Gamma_{jj}^2\,\mathrm{Var}(F_{ij}) \leq \Gamma_{\max}^4\overline{V}.$$

By standard results on the expected spectral norm of symmetric random matrices with independent entries (Bai and Yin, 1988), we have that $\|\boldsymbol{H}\|] = O_p(\sqrt{n})$. Under our assumption that $\boldsymbol{F}$ has uniformly bounded support, this can be extended (Dallaporta, 2012, Theorem 2) to the statement that $\mathbb{E}[\|\boldsymbol{H}\|] = O(\sqrt{n})$.

Second term: $\mathbb{E}[\|\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|]$**.** Since $\boldsymbol{\Phi} = \widehat{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}$ and $\widehat{\Gamma}_{ii} = 1/\sqrt{\widehat{D}_{ii}}$, we use a Taylor expansion around D_{ii} to approximate[39] Φ_{ii}:

$$\Phi_{ii} = \widehat{\Gamma}_{ii} - \Gamma_{ii} = -\frac{1}{2}\Gamma_{ii}^3 G_{ii} + O(G_{ii}^2).$$

The variance of Φ_{ii} satisfies:

$$\mathrm{Var}(\Phi_{ii}) \leq \left(\frac{1}{2}\Gamma_{\max}^3\right)^2 \mathrm{Var}(G_{ii}) \leq \frac{C}{n} \tag{26}$$

for some constant $C > 0$. The entries of $\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}$ are:

$$(\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi})_{ij} = \Gamma_{ii}\underline{D}_{ij}\Phi_{jj}.$$

Therefore, the squared Frobenius norm is:

$$\|\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|_F^2 = \sum_{i,j}(\Gamma_{ii}\underline{D}_{ij}\Phi_{jj})^2.$$

Taking expectations:

$$\mathbb{E}[\|\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|_F^2] \leq \Gamma_{\max}^2(\underline{D}_{\max})^2 \sum_{i,j}\mathbb{E}[\Phi_{jj}^2] \leq Cn^2 \cdot \frac{1}{n} = Cn,$$

where in the penultimate step we have used eq. (26). Thus, using the fact that Frobenius norm bounds the spectral norm and Jensen's inequality, we have:

$$\mathbb{E}[\|\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|] \leq \mathbb{E}[\|\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|_F] \leq \sqrt{\mathbb{E}[\|\boldsymbol{\Gamma}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|_F^2]} \leq C\sqrt{n}.$$

Third term: $\mathbb{E}[\|\boldsymbol{\Gamma}\, \boldsymbol{F}\, \boldsymbol{\Phi}\|]$**.** The entries of $\boldsymbol{\Gamma}\, \boldsymbol{F}\, \boldsymbol{\Phi}$ are:

$$(\boldsymbol{\Gamma}\, \boldsymbol{F}\, \boldsymbol{\Phi})_{ij} = \Gamma_{ii}F_{ij}\Phi_{jj}.$$

The squared Frobenius norm is:

$$\|\boldsymbol{\Gamma}\, \boldsymbol{F}\, \boldsymbol{\Phi}\|_F^2 = \sum_{i,j}(\Gamma_{ii}F_{ij}\Phi_{jj})^2.$$

Taking expectations and using the Cauchy–Schwarz inequality:

$$\mathbb{E}[\|\boldsymbol{\Gamma}\, \boldsymbol{F}\, \boldsymbol{\Phi}\|_F^2] \leq \Gamma_{\max}^2\overline{V}\sum_{i,j}\mathbb{E}[\Phi_{jj}^2] \leq Cn^2 \cdot \frac{1}{n} = Cn.$$

Therefore:

$$\mathbb{E}[\|\boldsymbol{\Gamma}\, \boldsymbol{F}\, \boldsymbol{\Phi}\|] \leq C\sqrt{n}.$$

Fourth term: $\mathbb{E}[\|\boldsymbol{\Phi}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|]$ The entries of $\boldsymbol{\Phi}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}$ are:

$$(\boldsymbol{\Phi}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi})_{ij} = \Phi_{ii}\underline{D}_{ij}\Phi_{jj}.$$

The squared Frobenius norm is:

$$\|\boldsymbol{\Phi}\, \underline{\boldsymbol{D}}\, \boldsymbol{\Phi}\|_F^2 = \sum_{i,j}(\Phi_{ii}\underline{D}_{ij}\Phi_{jj})^2.$$

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[39]Let $f(x) = 1/\sqrt{x}$. The derivative is $f'(x) = -\frac{1}{2}x^{-3/2}$. Expanding $f(D_{ii} + G_{ii})$ around $x = D_{ii}$ using Taylor's theorem, we obtain $f(D_{ii} + G_{ii}) = f(D_{ii}) + f'(D_{ii})G_{ii} + O(G_{ii}^2)$. Therefore, $\Phi_{ii} = \widehat{\Gamma}_{ii} - \Gamma_{ii} = f(D_{ii} + G_{ii}) - f(D_{ii}) = f'(D_{ii})G_{ii} + O(G_{ii}^2) = -\frac{1}{2}\Gamma_{ii}^3 G_{ii} + O(G_{ii}^2)$, where $\Gamma_{ii} = f(D_{ii})$.

Taking expectations:

$$\mathbb{E}[\|\mathbf{\Phi}\,\underline{\boldsymbol{D}}\,\mathbf{\Phi}\|_F^2] \leq (\underline{D}_{\max})^2 \sum_{i,j} \mathbb{E}[\Phi_{ii}^2]\mathbb{E}[\Phi_{jj}^2] \leq Cn^2 \left(\frac{C}{n}\right)^2 = C.$$

Thus:

$$\mathbb{E}[\|\mathbf{\Phi}\,\underline{\boldsymbol{D}}\,\mathbf{\Phi}\|] \leq C.$$

Fifth term: $\mathbb{E}[\|\mathbf{\Phi}\,\boldsymbol{F}\,\mathbf{\Phi}\|]$ The entries of $\mathbf{\Phi}\,\boldsymbol{F}\,\mathbf{\Phi}$ are:

$$(\mathbf{\Phi}\,\boldsymbol{F}\,\mathbf{\Phi})_{ij} = \Phi_{ii}F_{ij}\Phi_{jj}.$$

The squared Frobenius norm is:

$$\|\mathbf{\Phi}\,\boldsymbol{F}\,\mathbf{\Phi}\|_F^2 = \sum_{i,j}(\Phi_{ii}F_{ij}\Phi_{jj})^2.$$

Taking expectations and using the Cauchy–Schwarz inequality,

$$\mathbb{E}[\|\mathbf{\Phi}\,\boldsymbol{F}\,\mathbf{\Phi}\|_F^2] \leq \overline{V} \sum_{i,j} \mathbb{E}[\Phi_{ii}^2]\mathbb{E}[\Phi_{jj}^2] \leq Cn^2 \left(\frac{C}{n}\right)^2 = \frac{C}{n}.$$

Therefore:

$$\mathbb{E}[\|\mathbf{\Phi}\,\boldsymbol{F}\,\mathbf{\Phi}\|] \leq \frac{C}{\sqrt{n}}.$$

Combining all the bounds:

$$\mathbb{E}[\|\boldsymbol{E}\|] \leq C_1\sqrt{n} + 2C_2\sqrt{n} + 2C_3\sqrt{n} + C_4 + \frac{C_5}{\sqrt{n}} = O(\sqrt{n}),$$

where C_1, C_2, C_3, C_4, C_5 are constants independent of n, as desired. □