The Power of Two in Token Systems

Itai Ashlagi
Department of Management Science and Engineering, Stanford University, Stanford, California 94305, iashlagi@stanford.edu

Süleyman Kerimov
Jones Graduate School of Business, Rice University, Houston, Texas 77005, kerimov@rice.edu

Omer Tamuz
Division of the Humanities and Social Sciences, Caltech, Pasadena, California 91125, tamuz@caltech.edu

In economies without monetary transfers, token systems serve as an alternative to sustain cooperation, alleviate free riding, and increase efficiency. This paper studies whether a token-based economy can be effective in marketplaces with thin exogenous supply. We consider a marketplace in which at each time period one agent requests a service, one agent provides the service, and one token (artificial currency) is used to pay for service provision. The number of tokens each agent has represents the difference between the amount of service provisions and service requests by the agent. We are interested in the behavior of this economy when very few agents are available to provide the requested service. Since balancing the number of tokens across agents is key to sustain cooperation, the agent with the minimum amount of tokens is selected to provide service among the available agents. When exactly one random agent is available to provide service, we show that the token distribution is unstable. However, already when just two random agents are available to provide service, the token distribution is stable, in the sense that agents' token balance is unlikely to deviate much from their initial endowment, and agents return to their initial endowment in finite expected time. Our results mirror the power of two choices paradigm in load balancing problems. Supported by numerical simulations using kidney exchange data, our findings suggest that token systems may generate efficient outcomes in kidney exchange marketplaces by sustaining cooperation between hospitals.

Key words: token systems; the power of two choices; kidney exchange

1. Introduction

Token systems have been introduced as a market solution to economies in which monetary transfers are undesirable or repugnant. Examples include trading favors in babysitting cooperatives (Sweeney and Sweeney 1977), exchanging resources in peer-to-peer systems (Vishnumurthy et al. 2003), and distributing food to food banks (Prendergast 2016). The use of such artificial currency is intended to sustain cooperation, alleviate free riding, and increase efficiency. The incentive for agents to provide service (or resources) is to earn tokens and the ability to spend them in future exchanges. This ability relies on the liquidity of agents' availability to provide service upon request. This paper studies the behavior of token systems in thin marketplaces, where demand exceeds supply, and agents' availability for transactions is sparse.

The motivation for this study arises from kidney exchange marketplaces. Platforms for kidney exchange emerged to address demand by incompatible patient-donor pairs who are seeking a kidney transplant (Roth et al. 2007). Token systems have been adopted by several platforms in order to increase cooperation between hospitals and alleviate free riding behavior (Ashlagi et al. 2013). Kidney exchanges are inherently thin due to the requirements for compatibility and the prevalence of many hard-to-match pairs. We ask if a token system can be effective in such a sparse marketplace.

To address this question, we study a stylized model in which agents request and provide service over time, and tokens are used as artificial currency to pay for service provision. We consider an infinite horizon model with finitely many agents, where each agent has initially 0 tokens, and agents are allowed to have negative number of tokens. At each time period, one randomly chosen agent requests a service, and a random subset of agents of size d become available to provide the service. One of these agents is selected to provide the requested service, and the service requester pays one token to the service provider. The number of tokens each agent has represents the difference between the amount of service provisions and service requests by the agent. As balancing the number of tokens across agents is key to sustain cooperation, the agent with the fewest tokens among the available agents is selected to provide the service.

We are interested in the case in which d is a small constant; in the context of kidney exchange, this aims to capture few match opportunities for a given patient-donor pair. Our model is based on Johnson et al. (2014) and Kash et al. (2015). Their models, however, assume that at each time period, a constant fraction of agents are available to provide service, i.e., the service availability is not minimal. Intuitively, the larger the number of agents who are available to provide service, the easier it is to balance the amount of service provisions among agents.

Cooperation is important for token systems. For example, the Capitol Hill Babysitting Co-op, which aimed to exchange babysitting hours between families has crashed, since tokens' values depreciated (Sweeney and Sweeney 1977). So in a healthy market, agents should not accumulate too many tokens (which can lead to unraveling), or accumulate a large debt (which leads to free riding).

Motivated by these potential frictions, we analyze the token distribution in our model, and identify conditions under which the token system is *stable*, in the sense that the Markov chain describing the process admits a stationary probability distribution (the formal definition is given in §2). Informally, stability is described by two desired conditions. The first condition is a uniform boundedness condition; the number of tokens each agent has does not deviate much from its initial state, with high probability. The second condition is a fairness condition; the number of tokens each agent has oscillates in such a way that agents return to their initial endowment in finite expected time. In other words, the first condition ensures that agents will not accumulate or lose too many tokens. Thus, agents will not lose their incentive to cooperate. The second condition implies that the market clears in finite expected time, which alleviates free riding and balances the number of requests and provisions for each agent continuously over time.

Overview of results. In the baseline case ($d = 1$), only one agent is available to provide service. Thus at each time period, service requester and provider are chosen independently to exchange service for tokens. In this case the system is unstable, since the number of tokens each agent has behaves like a divergent or null recurrent random walk.

When $d > 1$, the service provider who has the minimal number of tokens is chosen among all available agents. In our main finding, we show that already when $d = 2$, the system is stable. Moreover, we show that the long-run probability of agents having more than M tokens (or less than $-M$ tokens) is at most $\mathcal{O}(1/M)$. We further show that when the number of agents is $n = 2$ or as n grows large, this probability is exponentially small in M. We conjecture that this probability is exponentially small in M for any $n \geq 2$.

We also perform numerical experiments to simulate the token distribution of participating hospitals using data from the National Kidney Registry (NKR) platform. The simulations reveal and validate that easy-to-match pairs have, on average, more than one, but very few compatible hard-to-match pairs. Despite this sparsity, hospitals' tokens do not deviate much from the initial state, which is aligned with our predictions. It is worth noting that hospitals in our data vary significantly with respect to size and distribution of patient-donor pair characteristics.

Techniques. This paper is inspired and borrows from the literature on the power of two choices (see., e.g., Mitzenmacher 1996, Vvedenskaya et al. 1996, Azar et al. 1999). The main finding of this literature is that in load balancing problems, minimal choice can significantly reduce congestion both in dynamic and static settings. In our model, it is simple to analyze the system directly using a birth-death process when $n = 2$. For $n \geq 3$, the system does not seem amenable to a complete analytical solution, but softer techniques allow us to show that it is stable.

For large n, we use Kurtz's theorem on *density dependent Markov chains* (see Kurtz 1981) to analyze the token distribution; the same techniques are used in Mitzenmacher (1996) and Vvedenskaya et al. (1996) to analyze various load balancing problems including the supermarket model. While there are subtle differences between our model and the load balancing problems in the literature (as we explain below), our work can be viewed as another application of the power of two choices paradigm. Kurtz's theorem provides conditions under which a stochastic process can be approximated by a deterministic process in the limit. This allows us to show that the token distribution is balanced, and the number of tokens each agent has is unlikely to be far from its initial state.

Related literature. Numerous papers study exchange economies using tokens or models for exchanging favors (Möbius 2001, Friedman et al. 2006, Hauser and Hopenhayn 2008, Abdulkadiroglu and Bagwell 2012, Kash et al. 2012, 2015, Johnson et al. 2014). This literature is concerned with whether cooperation can be sustained in equilibrium, and whether efficiency can be achieved. The main finding of this literature is that if agents are sufficiently patient, token mechanisms may lead to efficient outcomes. Closely related to our paper are Johnson et al. (2014), Kash et al. (2015) and Bo et al. (2018). These papers study token systems as a strategic game in an infinite horizon with discounting, but with similar dynamics. The key difference is that these papers assume that either all or a constant fraction of agents are available to provide service at each time period. Kash et al. (2015) study a model in which the service provider is chosen independently from the token distribution, and show the existence of an equilibrium, in which agents provide service when their tokens is below some threshold. Johnson et al. (2014) study the same model and show that under the minimum token selection rule, agents always provide service in equilibrium when punishments are feasible. Bo et al. (2018) extend their findings without using punishments. We avoid game-theoretic modeling, and instead implicitly address the strategic environment using our stability conditions. Our paper contributes to this literature, by studying whether stability can be achieved with low liquidity (i.e., low availability of service).

Also related to our paper is the literature on the power of two choices in load balancing problems (Azar et al. 1994, 1999, Mitzenmacher 1996, Vvedenskaya et al. 1996). In this classic problem, n balls are sequentially thrown into n bins. The key finding is that if two bins are selected randomly and the ball is thrown to the bin with the lower load, then the fullest bin has exponentially fewer balls than if only one bin is chosen randomly in the throwing process. Mitzenmacher (1996) and Vvedenskaya et al. (1996) find a similar result for the supermarket model, which is a dynamic queueing system, where the longest queue is much longer if customers choose randomly among all queues rather than choosing intelligently between two random queues. The state space of the process we are interested in can be obtained by truncating the state space of the supermarket model. We describe in detail the difference between our model and the supermarket model in Remark 1 in §4.2. In short, using a queueing language, Mitzenmacher (1996) focuses on the effect of the parameter d on the expected time customers spend in the system and the length of the longest queue in the long-run. We are interested, however, in the overall distribution of all queue-lengths, and provide a more detailed characterization for the system, which is needed for our analysis. In fact, Azar et al. (1999) also study an infinite horizon process in which at each time period, one random ball is removed from the bins, and one ball arrives which is assigned to one of the two random bins intelligently. Our stochastic process has subtle differences (using this language, instead of removing one random ball, we first pick a random bin, and then remove a ball), making the machinery of Azar et al. (1999) inapplicable.

Notation. We use $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_+$ to denote the set of non-negative integers and the set of strictly positive integers, respectively. We use $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_+$ to denote the set of non-negative real numbers and the set of strictly positive real numbers, respectively. We write $\mathbb{E}_\pi[\cdot]$ and $\mathbb{P}_\pi(\cdot)$ to write the expectation and the probability with respect to a given distribution π, respectively.

2. Model

There is a finite set of agents $\mathcal{A} = \{1, 2, \dots, n\}$, $n \geq 2$. The time $t \in \mathbb{Z}_{\geq 0}$ is discrete. The number of tokens agent $i \in \mathcal{A}$ has at time t is denoted by $s_i^t \in \mathbb{Z}$. We assume that $s_i^0 = 0$ for all $i \in \mathcal{A}$. Let $s^t \in \mathbb{Z}^n$ track the number of tokens agents have at time t.

Let $P = (p_i)_{i\in\mathcal{A}}, Q = (q_i)_{i\in\mathcal{A}}$ be full-support probability measures over the set of agents. At each time period, nature picks one agent to become a *service requester* according to P. Let d be a positive integer, which we call *service availability density.* At each time period, nature picks *available service providers* by selecting d agents according to Q independently and with replacement. Thus, at most d agents are available to provide service at each time period. We say that agents are *symmetric* if $p_i = q_i = \frac{1}{n}$ for all $i \in \mathcal{A}$.

We refer the tuple (n, P, Q, d) as the token system. We will analyze the behavior of the token system under a natural matching policy called the *minimum token selection rule* (see, e.g., Johnson et al. 2014). This policy, at each time period, selects the available provider with the lowest number of tokens as the *service provider* (ties are broken by choosing uniformly at random). At each time t, if agent i is the service requester and agent j is the service provider, i pays one token to j. Note that an agent can provide service to herself.[1] In this case, $s^{t+1} = s^t$. Otherwise, $s_i^{t+1} = s_i^t - 1$, $s_j^{t+1} = s_j^t + 1$, and $s_k^{t+1} = s_k^t$ for all $k \in \mathcal{A}\backslash\{i, j\}$. In either case, we have $\sum_{i\in\mathcal{A}} s_i^t = 0$ for all $t \geq 0$. The case $d = 1$ is the degenerate case, where the system simply selects one service requester and one service provider independently at random.

Stability. Under a token system (n, P, Q, d), the state of amount of tokens s^t evolves according to a Markov chain defined on the state space $\left\{s \in \mathbb{Z}^n : \sum_{i\in\mathcal{A}} s_i = 0\right\}$. Our assumptions that P

[1] This assumption is motivated by the kidney exchange setting, where patient-donor pairs from the same hospital can be matched internally in a centralized setting. Our results hold with minor modifications if an agent cannot serve herself.

and Q are full-support probability measures over $\mathcal{A}$ ensure that this Markov chain is irreducible. Furthermore, since there is a positive probability that the service requester is the service provider herself at each time period, the Markov chain is aperiodic.

We say that a token system (n, P, Q, d) is *stable* if this Markov chain has a stationary probability distribution. The reason we associate the existence of a stationary distribution with stability is the fact that it is equivalent to each of the following conditions:

- **(C1)** There is a uniformly small probability that the number of tokens owned or owed by any agent is large. Formally, there is a function $f\colon \mathbb{Z}_{\geq 0} \to [0,1]$ such that $\lim_{M\to\infty} f(M) = 0$, and for all times t large enough and all agents $i \in \mathcal{A}$ it holds that $\mathbb{P}(|s_i^t| > M) < f(M)$.
- **(C2)** The expected time for the token system to clear is finite. Formally, let T_0 be the first time the system returns to 0, i.e., $T_0 = \min\{t > 0 : s^t = 0\}$. Then $\mathbb{E}(T_0)$ is finite. Note that by the Markov property, this is also the expected time to return to 0 after a later visit to 0.

While we do not incorporate strategic considerations in our model, we view stability as a necessary condition to sustain cooperation in an appropriately defined strategic game.[2] Since the chain is irreducible and aperiodic, stability implies that the stationary distribution is unique, and that over time the distribution of s^t will converge to it. Conversely, if there is no stationary distribution, the distribution of s^t will become more and more "spread out" as t increases, with some agents either owning or owing a large number of tokens. We thus interpret stability as a necessary condition for the prevention of unravelling and free riding in a token system.

The key novel feature of our model is that the service availability density d is small, rather than being a constant fraction of n. The larger the service availability density d is, the easier it is to achieve stability, as weakly more agents can provide service at any time period, which provides more flexibility to balance service provisions among agents. Therefore unless stated otherwise, we focus on the case when the service availability is minimal, i.e., $d = 2$.

Application: kidney exchange. We start with some background. Kidney exchange platforms arrange exchanges between incompatible patient-donor pairs so that patients swap their intended donors. Patient-donor pairs enrolled to these platforms are typically very hard or very easy to match, and a large fraction of them are hard-to-match (Ashlagi et al. 2013).[3] Naturally, this makes the compatibility graph of patient-donor pairs very sparse. In the United States, participation on these platforms is voluntary, so that hospitals decide whether and which incompatible pairs

[2] For example, in order to prove the existence of an equilibrium, where all agents play a threshold strategy, Kash et al. (2015) first determine the token distribution of agents in the long-run.

[3] Either due to their intended donor's blood type, or because the patient is highly sensitized.

to register to the platform.[4] A common behavior of hospitals is to match easy-to-match pairs to each other internally and enroll pairs that cannot be matched internally. Matching two easy-to-match pairs to each other rarely contributes to efficiency of the marketplace, given the large fraction of hard-to-match pairs (Ashlagi and Roth 2021). To overcome these frictions, the National Kidney Registry (NKR) adopted a token system that rewards hospitals based on their marginal contribution to the platform.[5] Matches, or exchanges, that are arranged through the platform generate a transfer of tokens between the hospitals and the platform (Agarwal et al. 2019).

Our stylized model can be applied here as follows. Hospitals can be viewed as the agents in our model. Keeping in mind that matching an easy-to-match pair to a hard-to-match pair contributes to efficiency the most, at each time period a (random) hospital requests a service to match their easy-to-match pair, where a random subset of hospitals are available to provide service; these hospitals have a compatible hard-to-match pair. Among those hospitals, the one with the minimum number of tokens is chosen to be matched (minimum token selection rule); so that the hospital whose easy-to-match pair is matched pays one token to the hospital with the hard-to-match pair. Since for each hospital, the number of tokens she has equals the difference between the number of service requests and the number of service provisions by the hospital, the token system favors the hospital who has the largest contribution to the platform to match their hard-to-match pairs. We also provide simulation results using data from the National Kidney Registry (NKR) in §5.2 under the minimum token selection rule, where hospitals' hard- and easy-to-match pairs ratios differ significantly.

Tracking the evolution of the underlying compatibility graph is highly intractable. Instead, our model abstracts away from the graph structure and even from counting the number of hard-to-match pairs hospitals have in the pool (our service availability distribution Q remains fixed). Despite this abstraction, our model can shed light on the predicted behavior of token systems for kidney exchange platforms. The symmetric case corresponds to the case where hospitals are of the same size and have the same balance between easy- and hard-to-match pairs. We further provide insights for the case in which hospitals differ in sizes and balances in §3, 5.1, and Appendix 7.5.

2.1. The case $d = 1$

Note that when $d = 1$, s_i^t is a lazy random walk in one dimension for all $i \in \mathcal{A}$, and so by standard arguments the token system is not stable. Moreover, s^t is a random walk on $\mathbb{Z}^{n-1}$ ($n-1$ dimensions), and hence, the Markov chain will not be recurrent by Pólya's Recurrence Theorem for all $n \geq 4$, so that the market will eventually stop clearing at all. Missing details are deferred to Appendix 7.1.

[4] Hospitals also enroll altruistic donors without any intended patient, who are willing to donate a kidney and expect nothing in return.

[5] This program is called the Center Liquidity Contribution program (see `http://www.kidneyregistry.org/docs/CLC_Guidelines.pdf`), inspired by Ashlagi and Roth (2014) who pointed out the need for a "frequent flyer" program.

3. Two agents

In this section, we analyze the token distribution when there are only 2 agents ($n = 2$). Although understanding the token distribution for this case is simple, the analysis will provide insights regarding the token distribution for the general case. Informally, the best concentration around the initial point is achieved when $n = 2$; as the number of agents increases, the "distance" of the token distribution from the initial point increases as well.

PROPOSITION 1. *For $d \geq 2$, the token system with 2 agents is stable if and only if $q_1^d < p_1$, $q_2^d < p_2$. In this case, let π be the the steady-state distribution of the Markov chain $(s^t : t \geq 0)$. Then for all $M \in \mathbb{Z}_+$, we have*

$$\mathbb{P}_\pi(|s_i^t| > M) = \frac{\left(\frac{p_2 q_1}{p_1 - q_1^d}\left(\frac{p_2 q_1^d}{p_1(1-q_1^d)}\right)^M + \frac{p_1 q_2}{p_2 - q_2^d}\left(\frac{p_1 q_2^d}{p_2(1-q_2^d)}\right)^M\right)}{\left(1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}\right)},$$

for $i = 1, 2$. Moreover, the expected time between two successive occurrences of the initial state $(0, 0)$ is given by $1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}$.

The proof is straightforward and given in Appendix 7.2.[6] One implication of Proposition 1 is that the probability of owning or owing a large number of tokens decays exponentially as $f(M) \approx a^M$, where the constant $a \in (0, 1)$ can be found following Remark 4 in Appendix 7.2.

Proposition 1 identifies the level of asymmetry that can be tolerated between service request and service provision (within and across agents) rates. Moving forward, we focus on the symmetric case. Note that for symmetric agents and $d \geq 2$, Proposition 1 implies that the system is stable. When furthermore $d = 2$, we get that

$$\mathbb{P}_\pi(|s_i^t| > M) = \frac{2}{3}\left(\frac{1}{3}\right)^M,$$

and that $\mathbb{E}(T_0) = 3$. In the general case, we will argue that $a = 1/3$ is the best rate one can hope for.

4. The general case

We analyze here the symmetric case with any number of agents. The results for the general case can be summarized as follows:

[6] We note that similar results hold even with less amount of service availability in the following sense. Suppose that at each time period, at most 2 agents are available to provide service with probability $0 < \beta < 1$, and only one agent is available to provide service with probability $1 - \beta$, independently. The analysis of this system is very similar and given in Appendix 7.2.

1. In §4.1, we show that stability holds for all $n \geq 2$ and $d \geq 2$. We further show that (C1) holds with $f(M) = 5/M$ for all $n \geq 2$ (Theorem 1). In particular, the probability that a given agent has a large number of tokens decays to zero in a rate that does not depend on the number of agents.

2. In §4.2, we show that as n grows large, (C1) holds with $f(M) = (1/2)^M$ (Theorem 2). The proof organization for Theorem 2 is as follows. We first present *density dependent Markov chains* and Kurtz's theorem. We then model our system as a density dependent Markov chain, and use Kurtz's theorem to characterize our system in the limit via a system of ordinary differential equations. Finally, we study the solution of this system to prove Theorem 2.

3. We conjecture that (C1) holds for any $n \geq 2$ with $f(M) = a^M$, where $a \in [1/3, 1/2]$ (Conjecture 1).

4.1. Stability

In general, it seems difficult to determine whether a given system (n, P, Q, d) is stable. Indeed, the results of the previous section show that already for $n = 2$, this can be highly sensitive to the precise values of P and Q. The next result shows that in the symmetric case stability is achieved for any $n \geq 2$, assuming $d \geq 2$.

THEOREM 1. *Assume that there are $n \geq 2$ symmetric agents. Then the system is stable for any $d \geq 2$. Furthermore, (C1) holds with $f(M) = 5/M$.*

Proof of Theorem 1. Denote by k^t the agent chosen to request service at time t, and denote by I^t the set of agents chosen to be available to provide service at time t. Let $|I^t|$ denote the size of I^t, and note that $|I^t|$ takes values in $\{1, 2, \ldots, d\}$. Let $j^t \in I^t$ be the agent chosen to provide service, i.e., an agent chosen uniformly from the agents in $i \in I^t$ that minimizes s_i^t. Hence $s_{j^t}^t = \min_{i \in I^t} s_i^t$, so that in particular, the service provider j^t is an agent with the minimum number of tokens among the available agents in I^t.

Let $V^t = \sum_{i=1}^n (s_i^t)^2$, and let

$$v^t := \mathbb{E}[V^t] = \sum_{i=1}^n \mathbb{E}[(s_i^t)^2].$$

Note that by symmetry, we have $v^t = n\mathbb{E}[(s_1^t)^2]$. Let E_t be the event $\{k^t = j^t\}$, i.e., the event that the service provider and requester are the same agent. Let E_t^c be the event $\{k^t \neq j^t\}$. Since k^t and j^t are independent and uniformly distributed, the probability of E_t is $1/n$. Note that conditioned on E^{t+1}, it holds that $s^{t+1} = s^t$. Then we have

$$\begin{aligned} v^{t+1} &= \frac{1}{n}\mathbb{E}[V^t | E_t] + \frac{n-1}{n}\mathbb{E}[V^t | E_t^c] \\ &= \frac{1}{n}\sum_{i \in \mathcal{A}} \mathbb{E}[(s_i^t)^2] + \frac{n-1}{n}\mathbb{E}[V^t | E_t^c] \\ &= \frac{1}{n}v^t + \frac{n-1}{n}\mathbb{E}[V^t | E_t^c]. \end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}[V^t|E_t^c] &= \sum_{i\in\mathcal{A}}\mathbb{E}[(s_i^{t+1})^2|E_t^c]\\ &= \mathbb{E}[(s_{k^t}^{t+1})^2|E_t^c]+\mathbb{E}[(s_{j^t}^{t+1})^2|E_t^c]+\sum_{i\in I^t\setminus\{j^t\}}\mathbb{E}[(s_i^{t+1})^2|E_t^c]+\sum_{i\in\mathcal{A}\setminus I^t\setminus\{k^t\}}\mathbb{E}[(s_i^{t+1})^2|E_t^c]\\ &= \mathbb{E}[(s_{k^t}^t-1)^2+(s_{j^t}^t+1)^2]+\sum_{i\in I^t\setminus\{j^t\}}\mathbb{E}[(s_i^t)^2]+\sum_{i\in\mathcal{A}\setminus I^t\setminus\{k^t\}}\mathbb{E}[(s_i^t)^2]\\ &= -2\mathbb{E}[s_{k^t}^t]+1+2\mathbb{E}[s_{j^t}^t]+1+\sum_{i\in\mathcal{A}}\mathbb{E}[(s_i^t)^2].\end{aligned}$$

Since $\mathbb{E}[s_i^t]=0$ for all $i\in\mathcal{A}$ and $t\geq 0$, and using the fact that k^t is independent of s^t, we have $\mathbb{E}[s_{k^t}^t]=0$. Hence

$$\mathbb{E}[V^t|E_t^c]=v^t+2\mathbb{E}[s_{j^t}^t]+2,$$

and so

$$v^{t+1}=v^t+\frac{2(n-1)}{n}\mathbb{E}[s_{j^t}^t]+\frac{2(n-1)}{n}.$$

Without loss of generality, assume that $I^t=\{1,2,\dots,|I^t|\}$. Now with probability $n^{-(d-1)}$, we have that $|I^t|=1$, in which case $I^t=\{j^t\}=\{1\}$. Hence $\mathbb{E}[s_{j^t}^t \mid |I^t|=1]=0$. With probability $1-n^{-(d-1)}$, we have that $|I_t|\geq 2$, in which case $s_{j^t}^t=\min\{s_1^t,s_2^t,\dots,s_{|I^t|}^t\}\leq\min\{s_1^t,s_2^t\}$. Using the fact that $\min\{a,b\}=\frac{1}{2}(a+b-|a-b|)$ for all $a,b\in\mathbb{R}$, we get $2\mathbb{E}[\min\{s_1^t,s_2^t\}]=-\mathbb{E}[|s_1^t-s_2^t|]$, and we get

$$\mathbb{E}[s_{j^t}^t \mid |I^t|\geq 2]\leq -\frac{1}{2}\mathbb{E}[|s_1^t-s_2^t|].$$

Thus,

$$\mathbb{E}[s_{j^t}^t \mid |I^t|\geq 2]\leq -\frac{1}{2}(1-n^{-(d-1)})\mathbb{E}[|s_1^t-s_2^t|],$$

and we get

$$v^{t+1}\leq v^t-\frac{n-1}{n}(1-n^{-(d-1)})\mathbb{E}[|s_1^t-s_2^t|]+2.$$

Now we make use of the following claim. For real random variables $Y,Z_1,\dots,Z_n$, we have

$$\mathbb{E}\left[\left|Y-\frac{1}{n}\sum_{i=1}^n Z_i\right|\right]\leq\frac{1}{n}\sum_{i=1}^n\mathbb{E}[|Y-Z_i|], \tag{1}$$

where the claim follows immediately from convexity and Jensen's inequality. Again by the symmetry of the problem, we have

$$\mathbb{E}[|s_1^t-s_2^t|]=\frac{1}{n-1}\sum_{i=1}^n\mathbb{E}[|s_1^t-s_i^t|].$$

Therefore by (1), we have

$$\mathbb{E}[|s_1^t - s_2^t|] \geq \frac{n}{n-1}\mathbb{E}\left[\left|s_1^t - \frac{1}{n}\sum_{i=1}^n s_i^t\right|\right] = \frac{n}{n-1}\mathbb{E}[|s_1^t|],$$

where the second equality follows from the fact that $\sum_{i=1}^n s_i^t = 0$ for all $t \geq 0$. Therefore, we have

$$v^{t+1} \leq v_t - (1-n^{-(d-1)})\mathbb{E}[|s_1^t|] + \frac{2(n-1)}{n} \leq v_t - (1-n^{-(d-1)})\mathbb{E}[|s_1^t|] + 2,$$

and via recursion, we get

$$v^t \leq 2t - (1-n^{-(d-1)})\sum_{u=0}^{t-1}\mathbb{E}[|s_1^u|].$$

Since $v^t \geq 0$, we have

$$\frac{1}{t}\sum_{u=0}^{t-1}\mathbb{E}[|s_1^u|] \leq \frac{2}{1-n^{-(d-1)}} \leq 4, \tag{2}$$

where the second inequality follows from the fact that for $d,n \geq 2$, the denominator $1-n^{-(d-1)}$ is at least $1/2$.

Denote the distribution of s^t by $\mu^t \in \Delta(\mathbb{Z}^n)$, and let $\nu^t = \frac{1}{t}\sum_{u=0}^{t-1}\mu_t$. Let Y^t be a random variable with distribution ν^t for all $t \geq 1$, and denote by Y_i^t the i'th index of Y^t for all $i \in \mathcal{A}$. Then per (2), we have

$$\mathbb{E}[|Y_i^t|] = \frac{1}{t}\sum_{u=0}^{t-1}\mathbb{E}[|s_i^u|] \leq 4, \tag{3}$$

for all $i \in \mathcal{A}$.

Suppose towards a contradiction that the Markov chain has no stationary probability measure. Then for any finite subset $E \subset \mathbb{Z}^n$, it holds that $\lim_{t\to\infty}\mathbb{P}(s_i^t \in E) = 0$ by standard arguments about Markov chains. In particular, $\lim_{u\to\infty}\mathbb{E}[|s_i^u|] = \infty$, which contradicts (3). Thus, we have stability.

Since the Markov chain has a stationary distribution, and since it is irreducible and aperiodic, the distribution of s_i^t will converge to the stationary distribution. It follows from (3) that $\lim_{t\to\infty}\mathbb{E}[|s_i^t|] \leq 4$, and so $\mathbb{E}[|s_i^t|] \leq 5$ for all t large enough. Hence, it follows from Markov's inequality that for all t large enough and for all $i \in \mathcal{A}$, we have

$$\mathbb{P}(|s_i^t| \geq M) \leq \frac{5}{M}.$$

□

4.2. Exponential decay

Let π be the steady-state distribution of the Markov chain $(s^t : t \geq 0)$ granted by Theorem 1. For any $M \in \mathbb{Z}_+$, define $p_{n,M} := \mathbb{P}_\pi(|s_1^t| \leq M)$ when there are n symmetric agents.

THEOREM 2. $\lim_{n\to\infty} p_{n,M} \geq 1 - (1/2)^M$ *for all* $M \in \mathbb{Z}_+$.

We will use Kurtz's theorem on *density dependent Markov chains*, which allows us to analyze the stochastic process as a deterministic process in the limit. This analysis helps us to characterize the steady-state distribution of the system as the number of agents grows large and thus, proving Theorem 2.

Density dependent Markov chains. We begin with the definition of density dependent Markov chains, which is given in Kurtz (1981) for finite dimensional systems and extended by Mitzenmacher (1996) to countably infinite dimensional systems. Let $\mathbb{Z}^*$ be either $\mathbb{Z}^m$ for some finite dimension m, or $\mathbb{Z}^{\mathbb{N}}$, and similarly define $\mathbb{R}^*$. Let $L \subseteq \mathbb{Z}^*$ be the set of possible non-zero transitions of the system. For each $\vec{l} \in L$, define a nonnegative function $\beta_{\vec{l}}\colon \mathbb{R}^* \to [0,1]$.

DEFINITION 1. A sequence (indexed by n) of continuous time Markov chains $(X_n(t) : t \geq 0)$ on the state spaces $S_n = \left\{\vec{k}/n : \vec{k} \in \mathbb{Z}^*\right\}$ is a density dependent Markov chain if there exists a $\beta_{\vec{l}}\colon \mathbb{R}^* \to [0,1]$ such that for all n the transition rate of X_n is given by $q_{x,y}^{(n)} = n\beta_{n(y-x)}(x)$, $x, y \in S_n$.

In Definition 1, the index n can be interpreted as the total population or volume of the system, and the components of $\vec{k}/n$ can be interpreted as the densities of different types present in the system. The $\beta_{\vec{l}}(x)$ can be interpreted as the probability of transition $\vec{l}$ from $x \in S_n$ to $y \in S_n$, where $nx + \vec{l} = ny$. Given a density dependent Markov chain X_n with transition rates $q_{x,y}^{(n)} = q_{\vec{k},\vec{k}+\vec{l}}^{(n)} = n\beta_{\vec{l}}(\vec{k}/n)$, define $F(x) = \sum_{\vec{l}\in L} \vec{l}\beta_{\vec{l}}(x)$. The following theorem is key in our analysis:

THEOREM 3 **(Kurtz's theorem (Kurtz 1981, Mitzenmacher 1996))**. *Suppose we have a density dependent Markov chain* X_n *(of possibly countably infinite dimension) satisfying the Lipschitz condition* $|F(x) - F(y)| \leq M|x - y|$ *for some constant* M*. Further suppose* $\lim_{n\to\infty} X_n(0) = x_0$*, and let* X *be the deterministic process:*

$$X(t) = x_0 + \int_0^t F(X(u))du, \; t \geq 0. \tag{4}$$

Consider the path $\{X(u) : u \leq T\}$ *for some fixed* $T \geq 0$*, and assume that there exists a neighborhood* K *around this path satisfying*

$$\sum_{\vec{l}\in L} |\vec{l}| \sup_{x\in K} \beta_{\vec{l}}(x) < \infty. \tag{5}$$

Then $\lim_{n\to\infty} \sup_{u\leq T} |X_n(u) - X(u)| = 0$ *almost surely.*

The Lipschitz condition ensures the uniqueness of the solution for the differential equation $\dot{X} = F(X)$, which follows by taking the derivative of (4) with respect to t.[7] Condition (5) ensures that the jump rate is bounded in the process. Kurtz's theorem implies that as $n \to \infty$, the behavior of a density dependent Markov chain can be characterized by the deterministic process given in (4), where the convergence holds on a finite time interval $[0, T]$ for an arbitrary T. We next model and study our system as a density dependent Markov chain, which we refer to as the *finite model*.

The finite and infinite models. Let us model the system with n symmetric agents as a density dependent Markov chain and denote it by $(X_n(t), t \geq 0)$. Note that in Definition 1, the $\beta_{\vec{l}}$'s are independent of n, and the transition rates are linear in n. In order to fit our system to this definition, we assume that each agent has an exponential clock with rate 1. The ticking of agent i's clock corresponds to a service request by i, and the service provider is selected immediately using the minimum token selection rule. Note that because of the memoryless property and the continuity of the distribution that governs the clocks, agents request service uniformly, and exactly one agent requests service at a time. As a slight abuse of notation, let $n_i(t)$ be the number of agents with i tokens at time t, $m_i(t)$ be the number of agents with at least i tokens at time t, and $z_i(t) := m_i(t)/n$ be the fraction of agents with at least i tokens at time t. Let $\vec{z}(t) = (..., z_{-2}(t), z_{-1}(t), z_0(t), z_1(t), z_2(t), ...)$, and we drop the time index t when the meaning is clear. We represent the state of X_n by $\vec{z} = \vec{k}/n \in \mathbb{Z}^{\mathbb{N}}/n$. Let us call this process the *finite model*. Note that the initial state of X_n is $\vec{z}(0) = (..., 1, 1, 1, 0, 0, ...)$, where $z_i(0) = 1$ for all $i \leq 0$, and $z_i(0) = 0$ for all $i \geq 1$.

Next, we describe the transition probabilities $\beta_{\vec{l}}$'s. The set of possible non-zero transitions from $\vec{k} = n\vec{z}$ is $L = \{e_{ij} : i, j \in \mathbb{Z}, i \neq j\}$, where e_{ij} is an infinite dimensional vector of all zeros except the i'th index (which corresponds to the index of z_i) is -1 and the j'th index (which corresponds to the index of z_j) is 1. Note that after transition e_{ij} occurs, nz_i decreases by 1 and nz_j increases by 1 simultaneously. Hence, the transition e_{ij} corresponds to the event when an agent with i many tokens requests service and an agent with $j-1$ many tokens provides service. Since the probability that the service requester has i many tokens is $z_i - z_{i+1}$, and the probability that the service provider has j many tokens is $z_j^d - z_{j+1}^d$, we have $\beta_{e_{ij}}(\vec{z}) = (z_i - z_{i+1})(z_{j-1}^d - z_j^d)$.[8]

Denote the *infinite model* by X, which is the limit of the finite model X_n, i.e., $X = \lim_{n\to\infty} X_n$. Since X is characterized by the deterministic process (4), we need to analyze the components

[7] For example, see Lemma 4.1.6 in Abraham et al. (2012).

[8] The probability that all available agents have at least j many tokens is z_j^d, and we subtract the probability that all available agents have at least $j+1$ many tokens, which is z_{j+1}^d.

of $F(x)$. Note that the i'th component of $F(x) = \sum_{\vec{l} \in L} \vec{l} \beta_{\vec{l}}(x)$ (which corresponds to z_i) is $\sum_{j \in \mathcal{A} \setminus \{i\}} (z_j - z_{j+1})(z_{i-1}^d - z_i^d) - \sum_{j \in \mathcal{A} \setminus \{i\}} (z_i - z_{i+1})(z_{j-1}^d - z_j^d)$, which simplifies to

$$(1 - z_i + z_{i+1})(z_{i-1}^d - z_i^d) - (1 - z_{i-1}^d + z_i^d)(z_i - z_{i+1}) = (z_{i-1}^d - z_i^d) - (z_i - z_{i+1}). \tag{6}$$

Remark 1. In the well-known supermarket model, customers arrive according to a Poisson process with rate λn, $\lambda < 1$, where there are n servers that serve according to the FIFO (first in, first out) rule. An arriving customer considers only a constant number (d) of servers independently and uniformly at random from the n servers with replacement, and she joins the queue that contains fewest customers (any ties are broken arbitrarily). The service time for each customer is distributed exponentially with rate 1. Using this language, our model corresponds to this dynamic system, where the total number of customers is fixed throughout the process, which restricts the state space of the underlying process drastically.

Towards the proof of Theorem 2. Now that we have represented our system using the finite model, we are ready to prove Theorem 2. The proof is organized as follows. We first show that the conditions of Theorem 3 hold. Then using Theorem 3, we obtain the system of ordinary differential equations that characterize the infinite model. This characterization lets us represent the probability of interest in (C1) as n grows large using π_0 (the fraction of agents that have at least 0 tokens in the long-run). Finally, we find lower and upper bounds for π_0 to conclude.

Condition (5) is clearly satisfied since the magnitude of any jump is bounded, and the jump rate is bounded above by 1 for any state. We also show in Appendix 7.3 that the Lipschitz condition of Theorem 3 holds with $M = 2 + 2d$. By differentiating (4) with respect to t and using (6), we get the following system of ordinary differential equations that characterizes the infinite model:

$$\frac{dz_i}{dt} = (z_{i-1}^d - z_i^d) - (z_i - z_{i+1}) \text{ for all } i \in \mathbb{Z}. \tag{7}$$

Intuitively, (7) can be interpreted as follows. Let us consider the expected change in m_i (the number of agents with at least i many tokens) over a small time interval dt. First note that a transition occurs whenever one of the agent's exponential clock ticks, which happens with rate ndt. Under such transition, m_i increases by 1 if an agent with $i-1$ many tokens is selected as the service provider, which happens with probability $z_{i-1}^d - z_i^d$. m_i decreases by 1 if an agent with i many tokens is selected as the service requester, which happens with probability $z_i - z_{i+1}$. Hence, the expected increase in m_i is $(z_{i-1}^d - z_i^d)ndt$, and the expected decrease in m_i is $(z_i - z_{i+1})ndt$, which gives $dm_i = (z_{i-1}^d - z_i^d)ndt - (z_i - z_{i+1})ndt$, and since $m_i/n = z_i$, dividing both sides by ndt gives (7).

Define an *equilibrium point*, which is a point $\vec{a}$ such that if $\vec{z}(t')=\vec{a}$, then $\vec{z}(t)=\vec{a}$ for all $t \geq t'$. Denote the equilibrium point of the infinite model by $\vec{\pi}$, and assume $d=2$ for simplicity from now on (the following arguments can be easily generalized for $d>2$). Clearly $\vec{\pi}$ is an equilibrium point of the infinite model if and only if $\frac{d\pi_i}{dt}=0$ for all $i \in \mathbb{Z}$. Moreover, since agents start with 0 tokens and exchange one token at each transition, the expected number of tokens agents have is 0, and it can be written as follows:

$$\sum_{i \in Z} i \cdot \frac{n_i}{n} = \sum_{i \geq 1} i \cdot \frac{n_i}{n} + \sum_{i \leq 0} i \cdot \frac{n_i}{n} = \sum_{i \geq 1} \frac{m_i}{n} - \sum_{i \leq 0} \frac{n-m_i}{n} = \sum_{i \geq 1} z_i - \sum_{i \leq 0} (1-z_i) = 0. \tag{8}$$

Using (7) and (8), $\vec{\pi}$ can be found by solving the following system of equations:

$$(\pi_{i-1}^2 - \pi_i^2) - (\pi_i - \pi_{i+1}) = 0 \text{ for all } i \in \mathbb{Z}, \tag{9}$$

$$\sum_{i \geq 1} \pi_i - \sum_{i \leq 0} (1-\pi_i) = 0. \tag{10}$$

Note that (9) implies $\pi_{i+1} - \pi_i^2 = \pi_0 - \pi_{-1}^2$ for all $i \in \mathbb{Z}$. Since $\lim_{i\to\infty} \pi_i = 0$, we have $\pi_0 = \pi_{-1}^2$, and inductively we have the following relation:

$$\pi_{i+1} = \pi_i^2 \text{ for all } i \in \mathbb{Z}. \tag{11}$$

Using (11), (10) becomes $\sum_{i \geq 1} \pi_0^{2^i} - \sum_{i \geq 0} (1-\pi_0^{2^{-i}}) = 0$. Such series are known as *lacunary series*, where the function has no analytic continuation across its disc of convergence (see Hadamard's Gap Theorem). There is no closed form expression for such series to the best of our knowledge and thus, we are unable to find the equilibrium point explicitly. Note that in the long-run, the probability that $|s_i^t| \leq M$ for any $i \in \mathcal{A}$ is equal to $\pi_{-M} - \pi_{M+1}$. Using (11), proving that for all $M \in \mathbb{Z}_+$, $\lim_{n\to\infty} p_{n,M} \geq 1 - a^M$ for some $a \in (0,1)$, is equivalent to proving that the following inequalities hold:

$$\pi_0^{2^{-M}} - \pi_0^{2^{M+1}} \geq 1 - a^M \text{ for all } M \in \mathbb{Z}_+. \tag{12}$$

Lemma 1. *We have* $\frac{1}{2} < \pi_0 < \frac{3}{4}$.

The proof of Lemma 1 is given in Appendix 7.3, and now we use it to prove Theorem 2.

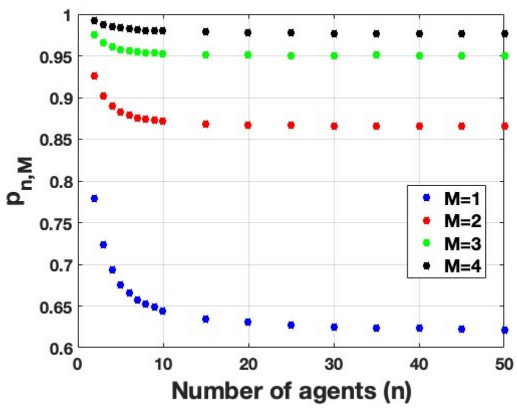
Proof of Theorem 2. We will show that $g(M) := \pi_0^{2^{-M}} - \pi_0^{2^{M+1}} - 1 + 2^{-M} \geq 0$ for all positive integers M, which implies that (12) is satisfied with $a = \frac{1}{2}$. Note that $\lim_{M\to\infty} g(M) = 0$. Hence, we will show that $g(M) \geq g(M+1)$ for all $M \in \mathbb{Z}_+$. It is easy to check that $g(M) \geq 0$ for all $M \leq 6$ using Lemma 1. The derivative of g with respect to M is $\frac{dg(M)}{dM} = \log(\pi_0) \cdot \log(0.5) \cdot \pi_0^{2^{-M}} \cdot 2^{-M} + \log(\pi_0) \cdot \log(0.5) \cdot \pi_0^{2^{M+1}} \cdot 2^{2M+1} \cdot 2^{-M} - \log(2) \cdot 2^{-M}$, where log is the natural logarithm. By Lemma 1, we have $\frac{1}{2} < \pi_0 < \frac{3}{4}$, and thus $0.19 < \log \pi_0 \cdot \log(0.5) < 0.5$. Since $\pi_0^{2^{-M}} \leq 1$, the first term in $\frac{dg(M)}{dM}$ is upper bounded by $\frac{1}{2} \cdot 2^{-M}$. For the second term, note that $\pi_0^8 \cdot 2 < \frac{1}{3}$. Since $2^{M+1} > 8(2M+1)$

for all $M \geq 6$, the second term is upper bounded by $\frac{1}{2} \cdot \frac{1}{3} \cdot 2^{-M}$, and $\frac{dg(M)}{dM}$ is upper bounded by $\frac{1}{2} \cdot 2^{-M} + \frac{1}{6} \cdot 2^{-M} - \log(2) \cdot 2^{-M}$, which is negative since $\log(2) > 2/3$. Hence, we have shown that $f(M)$ is a decreasing function on $[6, \infty]$, which concludes the proof. □

We close this section with the following conjecture:

CONJECTURE 1. *Assume that the agents are symmetric. Then for all $n \geq 2$, (C1) holds with $f(M) = a^M$ for all $M \in \mathbb{Z}_+$, for some $a \in [1/3, 1/2]$.*

Recall that $p_{n,M} = \mathbb{P}_\pi(|s_1^t| \leq M)$ when there are n symmetric agents, where π is the steady-state distribution of the Markov chain $(s^t : t \geq 0)$.[9] Figure 1 shows the behavior of $p_{n,M}$ and suggests the following conjecture:



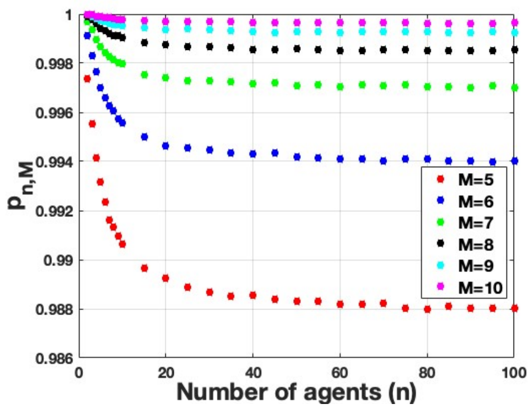


Figure 1 The behavior of $p_{n,M}$ versus n. In the simulation, the market runs until $t = 2 \cdot 10^7$ (the first $500,000$ time periods are ignored). For each n, the probabilities for each agent are computed separately, and the means are reported.

CONJECTURE 2. $p_{n+1,M} \leq p_{n,M}$ *for all $n \geq 2$ and for all $M \in \mathbb{Z}_+$.*

Note that Figure 1 suggests that for each M, $p_{n,M}$ converges to a limit point as n grows large. The precise values from the left figure for $n = 50$ are $p_{50,1} = 0.6184$, $p_{50,2} = 0.8645$, $p_{50,3} = 0.9500$ and $p_{50,4} = 0.9759$. Let $\pi_{-4} = \alpha$. Since $p_{\infty,M} = \pi_{-M} - \pi_{M+1}$, using (11), consider the following sets of equations and the corresponding positive real solutions:

- $p_{50,1} = \alpha^8 - \alpha^{64} = 0.6184$ with two positive real roots $\alpha \approx 0.947656, \mathbf{0.975067}$.
- $p_{50,2} = \alpha^4 - \alpha^{128} = 0.8645$ with two positive real roots $\alpha \approx 0.969503, \mathbf{0.975035}$.
- $p_{50,3} = \alpha^2 - \alpha^{256} = 0.9500$ with two positive real roots $\alpha \approx \mathbf{0.975598}, 0.984804$.
- $p_{50,4} = \alpha - \alpha^{512} = 0.9759$ with two positive real roots $\alpha \approx \mathbf{0.975904}, 0.991964$.

[9] Because of the symmetry, in the steady-state distribution, the probability of $-M \leq s_i^t \leq M$ equals to the probability of $-M \leq s_j^t \leq M$ for any agents $i, j \in \mathcal{A}$.

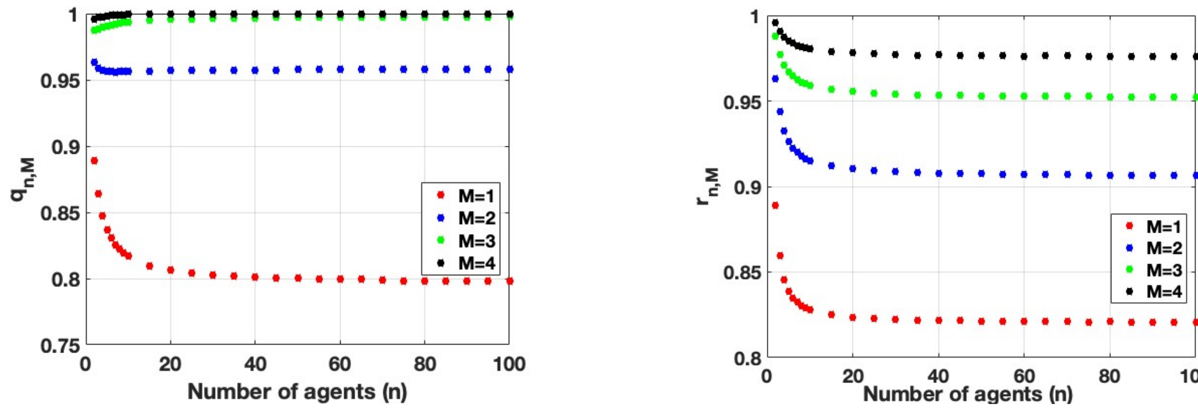


Figure 2 The behaviors of $q_{n,M}$ and $r_{n,M}$ versus n. In the simulations, the market runs until $t = 2 \cdot 10^7$ (the first $500,000$ time periods are ignored). For each n, the probabilities for each agent are computed separately, and the means are reported.

The closeness of the highlighted roots in the above sets of equations suggests that there is a consistency between the approximate limit point in Figure 1 and our analysis for the infinite model. Note that $\pi_{-4} \approx 0.975$ implies $\pi_0 \approx 0.667$ by (11). Using our analysis for the infinite model and assuming Conjecture 2, observe that (C1) is satisfied for the system with any number of symmetric agents with $\frac{1}{3} \leq a \leq \frac{1}{2}$, which implies Conjecture 1.

Remark 2. Given that there are n symmetric agents, let $q_{n,M} = \mathbb{P}(s_1^t \leq M)$ and $r_{n,M} = \mathbb{P}(s_1^t \geq -M)$, for all $M \in \mathbb{Z}_+$. Figure 2 shows the behaviors of $q_{n,M}$ and $r_{n.M}$. Interestingly, as Figure 2 shows, the monotonicity property seems to hold for $r_{n,M}$, but not for $q_{n,M}$.

5. Extensions and simulations

In this section, we first extend our main results for the symmetric case to a certain asymmetric environment. Then, we describe simulation results using data from the National Kidney Registry (NKR) platform.

5.1. $p_i = q_i$ case

We are interested here in a special case, where agents are asymmetric. We assume that there are $n \geq 2$ agents, and for every agent $i \in \mathcal{A}$, we have $p_i = q_i$, where $p_i, q_i \in \mathbb{Q}_+$ (we call this System I).

To analyze System I, we will first create a system (System II) with symmetric agents under similar transition dynamics that is equivalent to System I, where for each $i \in \mathcal{A}$, we create a group G_i with g_i agents such that $\frac{g_i}{g_j} = \frac{p_i}{p_j}$ for all $i, j \in \mathcal{A}$. Let $\sum_{i \in \mathcal{A}} g_i = N$. Finally, we consider an original system with N symmetric agents (System III). By showing that for each trajectory of System II, there exists an equivalent trajectory of System III, we obtain the following result (the details of the proof are delegated to Appendix 7.4):

Theorem 4. *Assume that there are $n \geq 2$ agents with $p_i = q_i$ for all $i \in \mathcal{A}$, where $p_i, q_i \in \mathbb{Q}_+$. Then the token system is stable for any $d \geq 2$.*

We also discuss a simple asymmetric case with two types of agents in Appendix 7.5 and derive analogous differential equations to the symmetric case, which can be used for numerical studies.

5.2. Simulations

In this section, we present results from numerical simulations using data from the National Kidney Registry (NKR) platform. We simulated the token distribution of participating hospitals under the minimum token selection rule. There are 1881 patient-donor pairs and 84 hospitals in the data. We restrict our attention only to exchanges between two patient-donor pairs (2-way cycles) and refer to such exchanges as matches. Each time period represents one day. All hospitals have 0 tokens at the beginning, and the kidney exchange pool is initially empty. At each time period, the following steps are conducted in order:

- ***Step 1****: Sample with replacement a patient-donor pair p uniformly at random from the entire set of pairs. This pair p is a tentative service requester.*
- ***Step 2****: Among the pairs that are waiting in the pool, identify the set of patient-donor pairs who can match with the pair p.*
  - ***Step 2.1****: If p has more than one possible match, use the minimum token selection rule to determine the service provider; that is, match p with the patient-donor pair that belongs to the hospital with the least amount of tokens (ties are broken uniformly at random). After the match is performed, the hospital of the service requester pays one token to the hospital of the service provider.*
  - ***Step 2.2****:: If p has no possible matches, add p to the pool (p is no longer a tentative service requester).*
- ***Step 3****: Each patient-donor pair in the pool leaves the system unmatched with probability $\frac{1}{365}$, independently.*

Observe that we do not categorize the easy- and hard-to-match pairs based on characteristics prior to the simulation. Instead, a pair is considered as a service requester (an easy-to-match pair) if it can match upon arrival to some pair in the pool. Figure 3 shows the token distribution of 84 hospitals after running the simulation 10^5 time periods (approximately 274 years). Note that overall, the token distribution is remarkably stable.

The are seven hospitals whose number of tokens drop below -30; the number of pairs in these hospitals are $1, 2, 2, 2, 2, 5$, and 28. The reason of deviation for the hospitals with only few pairs is apparent from the data; these hospitals only have easy-to-match pairs, and they match immediately upon arrival. The hospital with 28 pairs also has a large imbalance in favor of easy-to-match pairs

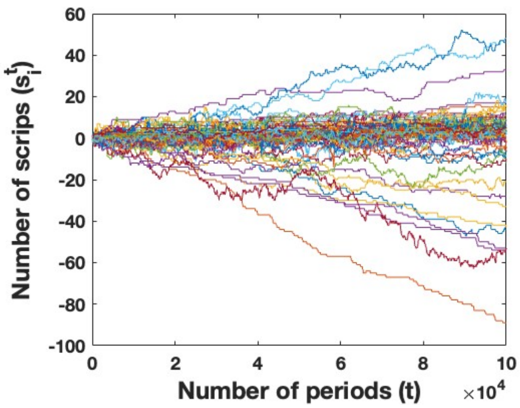


Figure 3 **The token distribution of 84 hospitals under the minimum token selection rule after 10^5 time periods.**

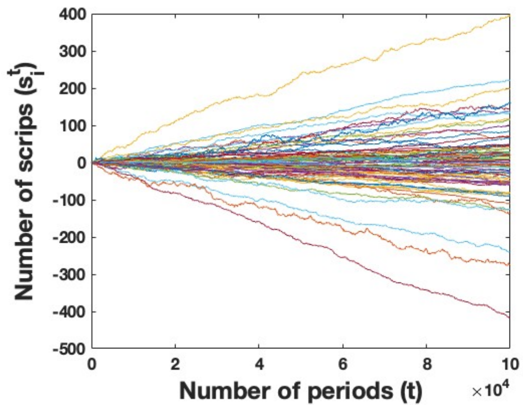


Figure 4 **The token distribution of 84 hospitals under the uniform selection rule after 10^5 time periods.**

(in contrast to the composition in typically large hospitals). This prevents the token system from rewarding back such hospitals. Three hospitals reach more than 30 tokens and have the following number of pairs: $6, 13$, and 18. The reason for the deviation in this case is that almost all of their pairs are hard-to-match. Figure 4 shows the token distribution of hospitals when the service provider is chosen uniformly at random, instead of using the minimum token selection rule. Note that there are large deviations and no oscillations.

Remark 3. The simulations also reveal that when there is at least one match in the pool for the service requester, almost 67% of the time there are at least two compatible pairs for the service requester; this suggests that there is often multiplicity of matchings (tie-breakings). However, the average number of matches for an easy-to-match pair (i.e., when a pair matches immediately upon arrival) is almost 7. This suggests that the average number of potential matches in the pool (or service availability) is small. So in this case, even few ties allow the token system to be stable.

6. Final remarks

This paper adapts methodologies from stochastic processes and ideas from the power of two choices literature to illustrate that token systems are likely to behave well even in thin marketplaces, where the availability of supply is very "little". We identified settings, under which the token system is stable when only few agents are available to provide service for any service request.

In the context of kidney exchange, token systems have been proposed and applied to incentivize participation by hospitals by accounting their contribution to the pool (Agarwal et al. 2019). Our findings suggest that the possibility of breaking ties in the matching process, even among few hospitals, enables the stability of the token system; breaking ties based on token balances allows to reward hospitals with their contribution and assures that the difference between hospitals' tokens remains small. Our numerical experiments reveal that tie-breakings are likely to happen in kidney exchange, and token systems can ensure cooperation between hospitals. It is interesting to expand this work to identify conditions under which dynamic token systems can be sustainable when the market exhibits more heterogeneity.

References

Abdulkadiroglu A, Bagwell K (2012) The optimal chips mechanism in a model of favors. Technical report, Working paper.

Abraham R, Marsden JE, Ratiu T (2012) *Manifolds, tensor analysis, and applications*, volume 75 (Springer Science & Business Media).

Agarwal N, Ashlagi I, Azevedo E, Featherstone CR, Karaduman Ö (2019) Market failure in kidney exchange. *American Economic Review* 109(11):4026–4070.

Ashlagi I, Jaillet P, Manshadi VH (2013) Kidney exchange in dynamic sparse heterogenous pools. *arXiv preprint arXiv:1301.3509* .

Ashlagi I, Roth AE (2014) Free riding and participation in large scale, multi-hospital kidney exchange. *Theoretical Economics* 9(3):817–863.

Ashlagi I, Roth AE (2021) Kidney exchange: An operations perspective. *Management Science* 67(9):5455–5478.

Azar Y, Broder AZ, Karlin AR (1994) On-line load balancing. *Theoretical Computer Science* 130(1):73–84.

Azar Y, Broder AZ, Karlin AR, Upfal E (1999) Balanced allocations. *SIAM journal on computing* 29(1):180–200.

Bo Y, Dawande M, Janakiraman G (2018) Analysis of scrip systems: On an open question in johnson et al.(2014). *Operations Research* .

Friedman EJ, Halpern JY, Kash I (2006) Efficiency and nash equilibria in a scrip system for p2p networks. *Proceedings of the 7th ACM conference on Electronic commerce*, 140–149 (ACM).

Hauser C, Hopenhayn H (2008) Trading favors: Optimal exchange and forgiveness. *Collegio Carlo Alberto Carlo Alberto Notebooks* 88.

Johnson K, Simchi-Levi D, Sun P (2014) Analyzing scrip systems. *Operations Research* 62(3):524–534.

Kash IA, Friedman EJ, Halpern JY (2012) Optimizing scrip systems: crashes, altruists, hoarders, sybils and collusion. *Distributed Computing* 25(5):335–357.

Kash IA, Friedman EJ, Halpern JY (2015) An equilibrium analysis of scrip systems. *ACM Transactions on Economics and Computation* 3(3):13.

Kurtz TG (1981) *Approximation of population processes*, volume 36 (SIAM).

Mitzenmacher MD (1996) *The power of two random choices in randomized load balancing*. Ph.D. thesis, PhD thesis, Graduate Division of the University of California at Berkley.

Möbius M (2001) Trading favors. *Mimeo, Microsoft Research* .

Prendergast C (2016) The allocation of food to food banks. *EAI Endorsed Trans. Serious Games* 3(10):e4.

Roth AE, Sönmez T, Ünver MU (2007) Efficient kidney exchange: Coincidence of wants in markets with compatibility-based preferences. *American Economic Review* 97(3):828–851.

Sweeney J, Sweeney RJ (1977) Monetary theory and the great capitol hill baby sitting co-op crisis: comment. *Journal of Money, Credit and Banking* 9(1):86–89.

Vishnumurthy V, Chandrakumar S, Sirer EG (2003) Karma: A secure economic framework for peer-to-peer resource sharing. *Workshop on Economics of Peer-to-Peer Systems*, volume 35.

Vvedenskaya ND, Dobrushin RL, Karpelevich FI (1996) Queueing system with selection of the shortest of two queues: An asymptotic approach. *Problemy Peredachi Informatsii* 32(1):20–34.

7. Appendix

7.1. Proofs from Section 2.1

We start with the proof of the following proposition regarding the case $d=1$, and then discuss about the transition dynamics.

Proposition 1. *The token system is not stable for any $n\geq 2, P>0$ and $Q>0$ when $d=1$.*

Proof of Proposition 1. Fix an agent $i\in\mathcal{A}$. Then s_i^t is a lazy random walk in one dimension. In particular, we have

$$s_i^{t+1}=\begin{cases} s_i^t+1 & \text{with probability} \sum_{j\in\mathcal{A}\setminus\{i\}} p_jq_i \\ s_i^t & \text{with probability } 1-\sum_{j\in\mathcal{A}\setminus\{i\}} p_jq_i-\sum_{j\in\mathcal{A}\setminus\{i\}} p_iq_j \\ s_i^t-1 & \text{with probability} \sum_{j\in\mathcal{A}\setminus\{i\}} p_iq_j \end{cases} \tag{13}$$

for all $t\geq 0$. Note that if $\sum_{j\in\mathcal{A}\setminus\{i\}} p_jq_i \neq \sum_{j\in\mathcal{A}\setminus\{i\}} p_iq_j$, then $\mathbb{E}[s_i^t]$ diverges as $t\to\infty$. Now assume that

$$\sum_{j\in\mathcal{A}\setminus\{i\}} p_jq_i=\sum_{j\in\mathcal{A}\setminus\{i\}} p_iq_j \text{ for all } i\in\mathcal{A}. \tag{14}$$

Following standard arguments, it readily follows that s_i^t will take all the values in $\mathbb{Z}$ with probability 1. Moreover, even though s_i^t will be 0 infinitely often with probability 1, the expected return time to 0 is infinity. Hence, (C1) and (C2) are not satisfied for any n, P and Q when $d=1$. □

Transition dynamics. For future calculations, let us denote the following outcome by $(i,(j,k))$: $i\in\mathcal{A}$ is the service requester, and $\{j,k\}\subset\mathcal{A}$ is the subset of agents who are available to provide service. At each time period t, the outcome $(i,(j,k))$ occurs with probability $p_i2q_jq_k$ if $j\neq k$ and with probability $p_iq_j^2$ if $j=k$. Hence, $i\in\mathcal{A}$ is the service requester at time t, $j\in\mathcal{A}$ is the service provider if $s_k^t>s_j^t$, $k\in\mathcal{A}$ is the service provider if $s_j^t>s_k^t$, and one of $j,k\in\mathcal{A}$ is selected as the service provider uniformly at random if $s_j^t=s_k^t$. Let r_{jk}^t, $j,k\in\mathcal{A}$, $j\neq k$, be the probability that given that only agents j and k are available to provide service at time t, agent j is the service provider. Then, we have

$$r_{jk}^t=\begin{cases} 1 & \text{if } s_k^t>s_j^t \\ \frac{1}{2} & \text{if } s_k^t=s_j^t \\ 0 & \text{if } s_k^t<s_j^t \end{cases}.$$

Fix $i\in\mathcal{A}$. We have $s_i^{t+1}=s_i^t+1$ if one of the following outcomes occurs at time t:

- $(j,(i,i))$, where $j\neq i$, which happens with probability $\sum_{j\in\mathcal{A}\setminus\{i\}} p_jq_i^2$.
- $(j,(i,k))$, where $j,k\neq i$, $s_k^t\geq s_i^t$ and agent i wins the tiebreak if any, which happens with probability $\sum_{j\in\mathcal{A}\setminus\{i\}}\sum_{k\in\mathcal{A}\setminus\{i\}} p_j2q_iq_kr_{ik}^t$.

Similarly, $s_i^{t+1}=s_i^t-1$ if one of the following outcomes occurs at time t:

- $(i,(j,j))$, where $j \neq i$, which happens with probability $\sum_{j\in\mathcal{A}\setminus\{i\}} p_i q_j^2$.
- $(i,(j,k))$, where $j,k \neq i$, $j\neq k$, which happens with probability $\sum_{j\in\mathcal{A}\setminus\{i\}}\sum_{k\in\mathcal{A}\setminus\{i,j\}} p_i q_j q_k$.
- $(i,(i,j))$, where $j\neq i$, $s_i^t \geq s_j^t$ and agent i loses the tiebreak if any, which happens with probability $\sum_{j\in\mathcal{A}\setminus\{i\}} p_i 2q_iq_jr_{ji}^t$.

Therefore, we have

$$s_i^{t+1} = \begin{cases} s_i^t+1 & \text{with probability} \sum_{j\in\mathcal{A}\setminus\{i\}} p_j q_i^2 + \sum_{j\in\mathcal{A}\setminus\{i\}}\sum_{k\in\mathcal{A}\setminus\{i\}} p_j 2q_iq_kr_{ik}^t \\ s_i^t-1 & \text{with probability} \sum_{j\in\mathcal{A}\setminus\{i\}} p_iq_j^2 + \sum_{j\in\mathcal{A}\setminus\{i\}}\sum_{k\in\mathcal{A}\setminus\{i,j\}} p_iq_jq_k + \sum_{j\in\mathcal{A}\setminus\{i\}} p_i2q_iq_jr_{ji}^t \\ s_i^t & \text{otherwise} \end{cases}. \tag{15}$$

Note that (15) behaves similar to (13); the main difference is that the transition probabilities in (15) change as t changes because of the r_{ij}^t terms, which are time-dependent. Processes such as (15) referred as heterogeneous random walks, where the transition probabilities are state-dependent.

7.2. Proofs from Section 3

We start with the proof of Proposition 1. First, we describe the model here for convenience. Assume that there are 2 agents ($n=2$). We analyze the following discrete time birth-death process with the state space $\mathcal{S} = \{(a,b): a\geq 1, a\in\mathbb{N}, b=1,2\}\cup\{(0,0)\}$, which captures the system: the state (a,b) represents the case in which agent b has $a>0$ tokens, and $(0,0)$ is the initial state. Denote this birth-death process by $(Z_t : t\geq 0)$. Let $X^t = \max_{i\in\mathcal{A}} s_i^t$ and $Y^t = \min_{i\in\mathcal{A}} s_i^t$.

Proposition 1. *The token system with 2 agents is stable if and only if $q_1^d < p_1$, $q_2^d < p_2$ and $d\geq 2$. Let π be the the steady-state distribution of the Markov chain $(s^t : t\geq 0)$. Then for all $M\in\mathbb{Z}_+$, we have*

$$\mathbb{P}_\pi(|s_i^t|\leq M) = 1 - \frac{\left(\frac{p_2q_1}{p_1-q_1^d}\left(\frac{p_2q_1^d}{p_1(1-q_1^d)}\right)^M + \frac{p_1q_2}{p_2-q_2^d}\left(\frac{p_1q_2^d}{p_2(1-q_2^d)}\right)^M\right)}{\left(1+\frac{p_2q_1}{p_1-q_1^d}+\frac{p_1q_2}{p_2-q_2^d}\right)},$$

for $i=1,2$. Moreover, the expected time between two successive occurrences of the initial state $(0,0)$ *is given by* $1+\frac{p_2q_1}{p_1-q_1^d}+\frac{p_1q_2}{p_2-q_2^d}$.

Proof of Proposition 1. The transition probabilities for the process $(Z_t : t\geq 0)$ are as follows:

$$\Pr(Z_{t+1}=(1,1)\mid Z_t=(0,0)) = p_2\left(\sum_{i=1}^d \frac{i}{d}\binom{d}{i} q_1^iq_2^{d-i}\right),$$
$$\Pr(Z_{t+1}=(0,0)\mid Z_t=(1,1)) = p_1(1-q_1^d),$$
$$\Pr(Z_{t+1}=(1,2)\mid Z_t=(0,0)) = p_1\left(\sum_{i=1}^d \frac{i}{d}\binom{d}{i} q_2^iq_1^{d-i}\right),$$
$$\Pr(Z_{t+1}=(0,0)\mid Z_t=(1,1)) = p_1(1-q_1^d),$$

$$\Pr(Z_{t+1}=(1,2)\mid Z_t=(0,0))=p_1\left(\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_2^iq_1^{d-i}\right),$$
$$\Pr(Z_{t+1}=(0,0)\mid Z_t=(1,2))=p_2(1-q_2^d),$$
$$\Pr(Z_{t+1}=(a+1,1)\mid Z_t=(a,1))=p_2q_1^d \text{ for all } a\geq 1,$$
$$\Pr(Z_{t+1}=(a+1,2)\mid Z_t=(a,2))=p_1q_2^d \text{ for all } a\geq 1,$$
$$\Pr(Z_{t+1}=(a-1,1)\mid Z_t=(a,1))=p_1(1-q_1^d) \text{ for all } a\geq 2,$$
$$\Pr(Z_{t+1}=(a-1,2)\mid Z_t=(a,2))=p_2(1-q_2^d) \text{ for all } a\geq 2.$$

Assume that the steady-state exists, and denote the steady-state vector by π. The detailed balance equations are:

$$\pi_{(0,0)}p_2\left(\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_1^iq_2^{d-i}\right)=\pi_{(1,1)}p_1(1-q_1^d), \tag{16}$$

$$\pi_{(0,0)}p_1\left(\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_2^iq_1^{d-i}\right)=\pi_{(1,2)}p_2(1-q_2^d), \tag{17}$$

$$\pi_{(a,1)}p_2q_1^d=\pi_{(a+1,1)}p_1(1-q_1^d) \text{ for all } a\geq 1, \tag{18}$$

$$\pi_{(a,2)}p_1q_2^d=\pi_{(a+1,2)}p_2(1-q_2^d) \text{ for all } a\geq 1, \tag{19}$$

$$\pi_{(0,0)}+\sum_{b=1}^{2}\sum_{a=1}^{\infty}\pi_{(a,b)}=1. \tag{20}$$

It follows by (18) and (19) that

$$\pi_{(a,1)}=\pi_{(1,1)}\left(\frac{p_2q_1^d}{p_1(1-q_1^d)}\right)^{a-1} \text{ for all } a\geq 1, \tag{21}$$

$$\pi_{(a,2)}=\pi_{(1,2)}\left(\frac{p_1q_2^d}{p_2(1-q_2^d)}\right)^{a-1} \text{ for all } a\geq 1. \tag{22}$$

Using (16), (17), (20), (21) and (22), first note that since the infinite geometric series in (20) must converge, the following are necessary and sufficient conditions for $(Z_t : t\geq 0)$ to have a steady-state:

$$\frac{p_2q_1^d}{p_1(1-q_1^d)}<1 \text{ or } q_1^d<p_1, \tag{23}$$

$$\frac{p_1q_2^d}{p_2(1-q_2^d)}<1 \text{ or } q_2^d<p_2. \tag{24}$$

Using (20), (23), and (24), we have

$$\pi_{(0,0)} + \pi_{(1,1)}\left(\sum_{a=1}^{\infty}\left(\frac{p_2 q_1^d}{p_1(1-q_1^d)}\right)^{a-1}\right) + \pi_{(1,2)}\left(\sum_{a=1}^{\infty}\left(\frac{p_1 q_2^d}{p_2(1-q_2^d)}\right)^{a-1}\right) = 1. \tag{25}$$

Using (16), (17) and (25), we have

$$\pi_{(0,0)} = \left(1 + \frac{p_2(\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_1^i q_2^{d-i})}{p_1 - q_1^d} + \frac{p_1(\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_2^i q_1^{d-i})}{p_2 - q_2^d}\right)^{-1}. \tag{26}$$

Note that $\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_1^i q_2^{d-i} = q_1\sum_{i=1}^{d}\binom{d-1}{i-1}q_1^{i-1}q_2^{d-i} = q_1(q_1+q_2)^{d-1} = q_1$. Similarly, $\sum_{i=1}^{d}\frac{i}{d}\binom{d}{i}q_2^i q_1^{d-i} = q_2$. Hence, (26) simplifies to

$$\pi_{(0,0)} = \left(1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}\right)^{-1}. \tag{27}$$

Once we have $\pi_{(0,0)}$ as a function of p_i's and q_i's, we can write all the steady-state probabilities as a function of p_i's and q_i's. Note that in the steady-state, X^t is bounded above by $M \in \mathbb{Z}_+$ with probability $1 - \sum_{a=M+1}^{\infty}\pi_{(a,1)} - \sum_{a=M+1}^{\infty}\pi_{(a,2)}$, which is equal to

$$1 - \frac{\left(\frac{p_2 q_1}{p_1 - q_1^d}\left(\frac{p_2 q_1^d}{p_1(1-q_1^d)}\right)^M + \frac{p_1 q_2}{p_2 - q_2^d}\left(\frac{p_1 q_2^d}{p_2(1-q_2^d)}\right)^M\right)}{\left(1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}\right)}. \tag{28}$$

Since $X^t + Y^t = 0$ for all $t \geq 0$, for all $M \in \mathbb{Z}_+$, $\mathbb{P}_\pi(|s_i^t| \leq M)$ is given by (28) for $i = 1, 2$.

It is a well-known fact that starting from a state (a,b), the expected time of the first occurrence of state (a,b) is $\frac{1}{\pi_{(a,b)}}$. Hence, starting from state $(0,0)$, the expected time of the first occurrence of state $(0,0)$ is

$$\frac{1}{\pi_{(0,0)}} = \left(1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}\right). \tag{29}$$

□

REMARK 4. $\mathbb{E}[t_2 - t_1]$ in (C2) follows from (29). The constant a in (C1) can be found using (28) as follows. Define

$$x = \frac{p_2 q_1^d}{p_1(1-q_1^d)},\ y = \frac{p_1 q_2^d}{p_2(1-q_2^d)},\ c_1 = \frac{\frac{p_2 q_1}{p_1 - q_1^d}}{1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}},\ \text{and } c_2 = \frac{\frac{p_1 q_2}{p_2 - q_2^d}}{1 + \frac{p_2 q_1}{p_1 - q_1^d} + \frac{p_1 q_2}{p_2 - q_2^d}}.$$

Then (28) becomes $1 - c_1 x^M - c_2 y^M$. We want to find $0 < a < 1$ such that $1 - c_1 x^M - c_2 y^M \geq 1 - a^M$, or $a^M \geq c_1 x^M + c_2 y^M$ for all $M \in \mathbb{Z}_+$. Note that $c_1, c_2 < 1$. Hence, a can be chosen to be $x + y$ if $x + y < 1$. Consider the case when $x + y > 1$. Let $z = \max\{x, y\}$. Then $c_1 x^M + c_2 y^M < 2z^M$. Clearly, there exists $M' \in \mathbb{Z}_+$ such that $2z^M < 1$ for all $M \geq M'$. Pick $0 < a_1 < 1$ such that $a_1 > z$ and

$a_1^{M'} > 2z^{M'}$. Consider the inequalities $a^M \geq c_1 x^M + c_2 y^M$ for all $M < M'$. Since there are finitely many inequalities, we can pick $0 < a_2 < 1$ such that these inequalities are satisfied. Thus in this case, a can be chosen to be $\max\{a_1, a_2\}$.

Intermediate availability. Suppose that at each time period, we have $d = 2$ with probability β and $d = 1$ with probability $1 - \beta$, independently, for some $\beta \in (0,1)$. We can capture this system with the same birth-death process $(Z_t : t \geq 0)$ with updated transition probabilities. Following similar calculations as in the proof of Proposition 1, it follows that the following are necessary and sufficient conditions for the process to have a steady-state:

$$\frac{\beta p_2 q_1^d + (1-\beta) p_2 q_1}{\beta p_1 (1 - q_1^d) + (1-\beta) p_1 q_2} < 1, \tag{30}$$

$$\frac{\beta p_1 q_2^d + (1-\beta) p_1 q_2}{\beta p_2 (1 - q_2^d) + (1-\beta) p_2 q_1} < 1. \tag{31}$$

It also follows that

$$\pi_{(0,0)} = \left(1 + \frac{p_2 q_1}{\beta(p_1 - q_1^d) + (1-\beta)(p_1 q_2 - p_2 q_1)} + \frac{p_1 q_2}{\beta(p_2 - q_2^d) + (1-\beta)(p_2 q_1 - p_1 q_2)}\right)^{-1},$$

and in the steady-state, X^t is bounded above by $M \in \mathbb{Z}_+$ with probability

$$1 - \pi_{(0,0)}(A + B), \tag{32}$$

where

$$A = \frac{p_2 q_1}{\beta(p_1 - q_1^d) + (1-\beta)(p_1 q_2 - p_2 q_1)} \left(\frac{\beta p_2 q_1^d + (1-\beta) p_2 q_1}{\beta p_1 (1 - q_1^d) + (1-\beta) p_1 q_2}\right)^M,$$

$$B = \frac{p_1 q_2}{\beta(p_2 - q_2^d) + (1-\beta)(p_2 q_1 - p_1 q_2)} \left(\frac{\beta p_1 q_2^d + (1-\beta) p_1 q_2}{\beta p_2 (1 - q_2^d) + (1-\beta) p_2 q_1}\right)^M.$$

Similarly, since $X^t + Y^t = 0$ for all $t \geq 0$, for all $M \in \mathbb{Z}_+$, $\mathbb{P}_\pi(|s_i^t| \leq M)$ is given by (32) for $i = 1, 2$. When agents are symmetric and $d = 2$, (32) becomes $1 - \frac{2}{2+\beta}(\frac{2-\beta}{2+\beta})^M$. As Figure 5 shows, even for small values of β, the token distribution is fairly balanced with high probability.

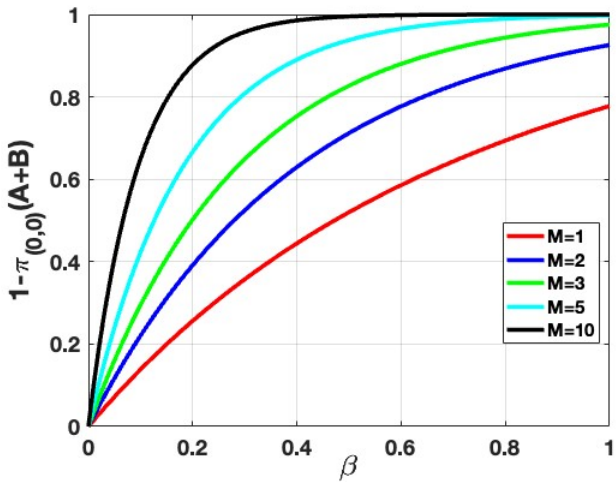


Figure 5 **(32) as a function of β for several values of M when the agents are symmetric and $d=2$.**

7.3. Proofs from Section 4

Proof of Lemma 1. We first prove that $\pi_0 > \frac{1}{2}$. Note that (10) can be written as

$$\sum_{M\geq 1}(\pi_M + \pi_{-M+1} - 1) = 0. \tag{33}$$

We claim that if $\pi_1 + \pi_0 = \pi_0^2 + \pi_0 - 1 < 0$, which implies $\pi_0 \leq \frac{1}{2}$, then all the terms in the summation (33) is negative, which is a contradiction. We have $\pi_M + \pi_{-M+1} - 1 = \pi_0^{2^M} + \pi_0^{2^{-M+1}} - 1$ by (11). Let $f(M) = 1 - \pi_0^{2^M} - \pi_0^{2^{-M+1}}$. Assume to the contrary that $\pi_0 \leq \frac{1}{2}$. Then $f(1) > 0$, and clearly we have $\lim_{M\to\infty} f(M) = 0$. We will show that $f(M) \geq f(M+1)$ for all $M \in \mathbb{Z}_+$. The derivative of $f(M)$ with respect to M is

$$\frac{df(M)}{dM} = -2^M \cdot \log(2) \cdot \pi_0^{2^M} \cdot \log(\pi_0) + 2^{-M+1} \cdot \log(2) \cdot \pi_0^{2^{-M+1}} \cdot \log(\pi_0), \tag{34}$$

where log is the natural logarithm. Since $\pi_0 \leq \frac{1}{2}$ by assumption, we have $\log(2) \cdot \log(\pi_0) < 0$. Thus, (34) has the same sign with

$$2^M \cdot \pi_0^{2^M} - 2^{-M+1} \cdot \pi_0^{2^{-M+1}} = \frac{2^{2M-1} \cdot \pi_0^{2^M} - \pi_0^{2^{-M+1}}}{2^{M-1}}. \tag{35}$$

Since $2^M > 2M - 1$ for all $M \geq 1$ and $2\pi_0 \leq 1$, we have

$$2^{2M-1} \cdot \pi_0^{2^M} - \pi_0^{2^{-M+1}} = (2\pi_0)^{2M-1} \cdot \pi_0^{2^M - 2M + 1} - \pi_0^{2^{-M+1}} \leq \pi_0^{2^M - 2M+1} - \pi_0^{2^{-M+1}} < 0 \tag{36}$$

for all $M > 1$, where the last inequality follows from the fact that $2^M - 2M + 1 > 0$, $-M + 1 < 0$, and $0 < \pi_0 < 1$.

Now we prove that $\pi_0 < \frac{3}{4}$. Assume to the contrary that $\pi_0 \geq \frac{3}{4}$. Then clearly we have

$$\sum_{i\geq 1}^{\infty} \pi_i = \sum_{i\geq 1}^{\infty} \pi_0^{2^i} \geq 0.8. \tag{37}$$

Now we want to find an upper bound for

$$\sum_{i\leq 0}^{\infty}(1-\pi_i)=\sum_{i\geq 0}^{\infty}(1-\pi_0^{2^{-i}}). \tag{38}$$

Since $\frac{1-\pi_0^{2^{-i}}}{1-\pi_0^{2^{-i-1}}}=1+\pi_0^{2^{-i-1}}$ is an increasing function for $i\geq 0$, we can upper bound (38) by the following geometric series

$$(1-\pi_0)+(1-\pi_0^{2^{-1}})+(1-\pi_0^{2^{-2}})(1+r+r^2+\ldots)\leq 0.6, \tag{39}$$

where $r=\frac{1-\pi_0^{2^{-2}}}{1-\pi_0^{2^{-3}}}$. But, (37) and (39) contradict to (10). $\square$

Lemma 2 **(Lipschitz condition)**. *The finite model satisfies the Lipschitz condition in L_1-distance.*

Proof of Lemma 2. Let $x=(x_i)_{i\in\mathbb{Z}}$ and $y=(y_i)_{i\in\mathbb{Z}}$ be two states of the finite model. By (6), we have

$$\begin{aligned}
|F(x)-F(y)| &= \sum_{i=-\infty}^{\infty}|(x_{i-1}^d-x_i^d)-(x_i-x_{i+1})-(y_{i-1}^d-y_i^d)+(y_i-y_{i+1})| \\
&\leq 2\sum_{i=-\infty}^{\infty}|x_i^d-y_i^d|+2\sum_{i=-\infty}^{\infty}|x_i-y_i| \\
&\leq (2+2d)\sum_{i=-\infty}^{\infty}|x_i-y_i| \\
&= (2+2d)|x-y|,
\end{aligned}$$

where in the first inequality we used the triangle inequality, and in the second inequality we used the expansion $(a-b)^n=(a-b)(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1})$ and the fact that $0\leq x_i,y_i\leq 1$ for all $i\in\mathbb{Z}$. $\square$

7.4. Proofs from Section 5

In this section we prove Theorem 4. Assume that there are n agents with $p_i=q_i$ for all $i\in\mathcal{A}^I$, where $p_i,q_i\in\mathbb{Q}_+$ (call it System I). Label the agents in System I by i^I where $i\in\{1,2,\ldots,n\}$. For simplicity, assume that $d=2$ (arguments can be easily generalized for general d). For all the systems, denote the outcome, in which agent i is the service requester, and agents j and k are available to provide service by $(i,(j,k))$, and assume that all agents start with 0 tokens. Consider the following system with the following transition dynamics (call it System II):

- *For each $i\in\mathcal{A}^I$, create a group G_i with g_i agents, where $\frac{g_i}{g_j}=\frac{p_i}{p_j}$ for all $i,j\in\mathcal{A}^I$. Denote the set of agents by $\mathcal{A}^{II}$. Let $\sum_{i\in\mathcal{A}^I}g_i=N$. Label the agents in System II by i^{II}, where $i\in\{1,2,\ldots,N\}$.*
- *At each time period, one agent requests service uniformly at random (with probability $\frac{1}{N}$).*

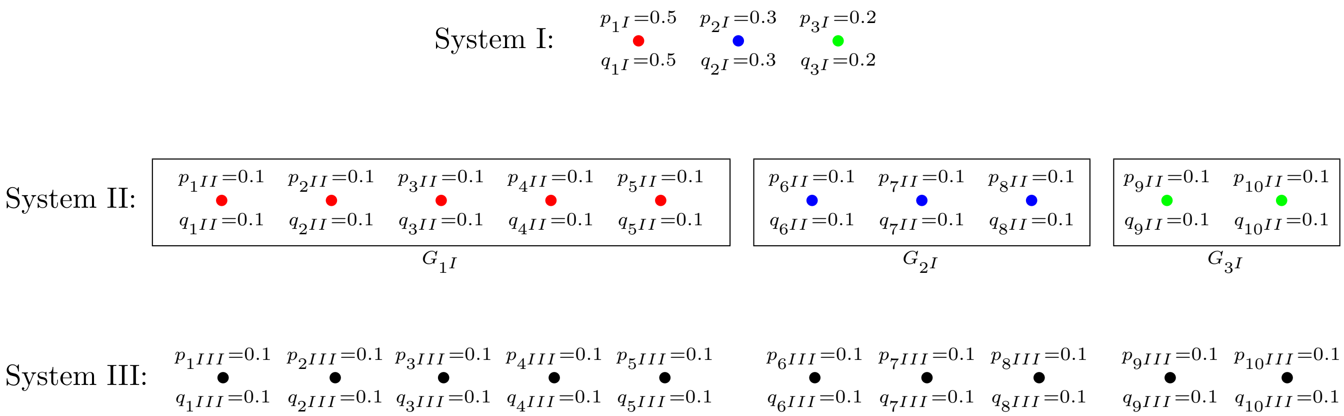


Figure 6 The representations of the systems described in Example 1.

- *At each time period, at most two agents are available to provide service. The available agents are chosen uniformly at random and with replacement.*
- *The service provider is chosen using the minimum token selection rule.*
- *Assume that agent $i^{II} \in \mathcal{A}^{II}$ is the service requester and agent $j^{II} \in \mathcal{A}^{II}$ is the service provider at time t. If i^{II} and j^{II} belong to the same group, then $s_k^{t+1} = s_k^t$ for all $k \in \mathcal{A}^{II}$. Otherwise, $s_k^{t+1} = s_k^t - 1$ for all k which is in the same group with i^{II}, $s_k^{t+1} = s_k^t + 1$ for all k which is in the same group with j^{II}, and $s_k^{t+1} = s_k^t$ for all other agents k.*

Note that System I and System II are equivalent processes. If $(i^I, (j^I, k^I))$ is realized for System I, the corresponding outcome for System II is $(i^{II}, (j^{II}, k^{II}))$, where $i^{II} \in G_{i^I}$, $j^{II} \in G_{j^I}$ and $k^{II} \in G_{k^I}$. Note that the outcomes have the same probabilities to occur. Under the corresponding outcomes, for any time t, we have $s_{i^I}^t = s_k^t$ for all $i^I \in \mathcal{A}^I$ and for all $k \in G_{i^I}$.

Finally, consider an original system with N symmetric agents (call it System III). Label the agents in System III by i^{III}, where $i \in \{1, 2, ..., N\}$, and consider the pairs (i^{II}, i^{III}) as couples. Similarly, if $(i^{II}, (j^{II}, k^{II}))$ is realized for System II, the corresponding outcome for System III is $(i^{III}, (j^{III}, k^{III}))$, and note that these outcomes occur with the same probability. We will compare the trajectories of System II and System III under the corresponding outcomes.

Example 1. Assume that there are three agents with $p_1 = q_1 = 0.5, p_2 = q_2 = 0.3$ and $p_3 = q_3 = 0.2$. Then System II consists of three groups G_1, G_2 and G_3, where $g_1 = 5, g_2 = 3$ and $g_3 = 2$. Hence, $N = 10$ and System III consists of 10 symmetric agents. The systems are shown in Figure 6.

Proof of Theorem 4. Start both System II and System III at the same time under the corresponding outcomes, and consider the sequence of outcomes up to some arbitrary time T. Fix a group G_{i^I} in System II. If under all the outcomes in this sequence the service providers are couples, then it is easy to see that for any $t \le T$, for any $u \in G_{i^I}$, we have $s_u^t = \sum_v s_v^t$, where the summation is over all agents $v \in \mathcal{A}^{III}$ such that v is a couple of some agent in G_{i^I}. Let T^* be the first time period when under the outcomes $(i^{II},(j^{II},k^{II}))$ and $(i^{III},(j^{III},k^{III}))$, the service providers are not couples. Then at time T^*-1, there must exist agents $j^*,k^*\in\mathcal{A}^{III}$ such that j^* has a couple which is in the same group with j^{II}, k^* has a couple which is in the same group with k^{II}, and at time T^*, under the outcomes $(i^{II},(j^{II},k^{II}))$ and $(i^{III},(j^*,k^*))$, the service providers are couples. Note that in System III, the probability that the outcome $(i^{III},(j^{III},k^{III}))$ is realized is the same as the probability that the outcome $(i^{III},(j^*,k^*))$ is realized. Thus, we have shown that for any trajectory $\mathcal{T}^{II}$ of System II, there exists a trajectory $\mathcal{T}^{III}$ of System III such that the service providers are couples throughout the processes. In other words, for each trajectory $\mathcal{T}^{II}$ of System II, there exists a trajectory of System III that captures $\mathcal{T}^{II}$. Clearly, the countable-state Markov chain which captures the token distribution of agents in System II with states $s^t\in\mathbb{Z}^N$ is irreducible and aperiodic. Using the above arguments, the positive recurrence of this chain follows from the fact that System III has a steady-state. This concludes the proof. $\square$

7.5. A simple asymmetric case

In this section, we further investigate whether the token system under the minimum token selection rule behaves well with asymmetric agents. In the context of kidney exchange, it is natural to ask whether large (or small) hospitals will have some advantage, or cause the system to be unstable. The system with asymmetric agents can also be modeled as a density dependent Markov chain. However, note that even in the symmetric case, there was no closed form expression for the equilibrium point. Thus, instead of finding a closed form expression for the equilibrium point, the analogous differential equations for the asymmetric case can be used for numerical studies.

We discuss a very simple asymmetric setting with two types of agents referred to as A and B, and let $d=2$ for simplicity. Assume that there are n agents where n is an even integer, and there are two types of agents: type A and type B. Agents within the same type have the same service requesting and providing rate. Assume that there are $n/2$ many type A agents and $n/2$ many type B agents for simplicity. Define the service requesting distribution by $P=(p_i)_{i\in\mathcal{A}}$, where $p_i=p_A$ if agent i is type A and $p_i=p_B$ if agent i is type B. This gives $\frac{n}{2}(p_A+p_B)=1$. Similarly, define the service availability distribution by $Q=(q_i)_{i\in\mathcal{A}}$, where $q_i=q_A$ if agent i is type A and $q_i=q_B$ if agent i is type B. This gives $\frac{n}{2}(q_A+q_B)=1$.

Similar to the finite model, in order to fit the system with asymmetric agents to the definition of a density dependent Markov chain, we can assume that each agent has an exponential clock with mean np_i. Ticking of agent i's clock corresponds to a service request by i. Note that the service requesting and service availability probabilities change as n changes, but the ratio between the probabilities do not change. Thus, let us assume that $p_B=\alpha p_A$ and $q_B=\beta q_A$ for some constants α and β which are independent of n.

Let $z_i^A(t)$ be the fraction of type A agents with at least i tokens at time t among the type A agents and $z_i^B(t)$ be the fraction of type B agents with at least i tokens at time t among the type B agents. Similar to the finite model, we will represent the state of the system by $\vec{z}(t) = (\vec{z^A}(t), \vec{z^B}(t))$ where $\vec{z^A}(t) = (..., z_{-1}^A(t), z_0^A(t), z_1^A(t), ...)$ and $\vec{z^B}(t) = (..., z_{-1}^B(t), z_0^B(t), z_1^B(t), ...)$. We drop the time index t when the meaning is clear. Note that the initial state of the system is $\vec{z}(0) = (\vec{z^A}(0), \vec{z^B}(0))$, where $z_i^A(0) = z_i^B(0) = 1$ for all $i \leq 0$, and $z_i^A(0) = z_i^B(0) = 0$ for all $i \geq 1$.

We will denote the set of possible transitions from $\vec{k} = \frac{n\vec{z}}{2}$ by $L = \{e_{ij}^A, e_{ij}^B, (e_i, -e_j), (-e_i, e_j) : i, j \in \mathbb{Z}, i \neq j\}$, where $e_{ij}^A = (e_{ij}, \vec{0})$ and $e_{ij}^B = (\vec{0}, e_{ij})$. Here, e_{ij} is an infinite dimensional vector of all zeros except the i'th index (which corresponds to the index of $z_i^A(t)$ or $z_i^B(t)$) is -1 and the j'th index (which corresponds to the index of $z_j^A(t)$ or $z_j^B(t)$) is 1, e_i is an infinite dimensional vector of all zeros except the i'th index (which corresponds to the index of $z_i^A(t)$ or $z_i^B(t)$) is -1. For example, e_{ij}^A corresponds to the transition where the service requester is a type A agent with i many tokens and the service provider is a type A agent with $j-1$ many tokens. Using these notations, we can compute the following probabilities:

- The probability that the service requester is a type A agent with i many tokens is $\frac{1}{1+\alpha}(z_i^A - z_{i+1}^A)$. Denote this probability by c_i^A.
- The probability that the service requester is a type B agent with i many tokens is $\frac{\alpha}{1+\alpha}(z_i^B - z_{i+1}^B)$. Denote this probability by c_i^B.
- The probability that the service provider is a type A agent with $j-1$ many tokens is $(\frac{1}{1+\beta})^2((z_{j-1}^A)^2 - (z_j^A)^2) + 2(\frac{1}{1+\beta})(\frac{\beta}{1+\beta})(z_{j-1}^A - z_j^A)z_j^B + 2(\frac{1}{1+\beta})(\frac{\beta}{1+\beta})(z_{j-1}^A - z_j^A)(z_{j-1}^B - z_j^B)\frac{1}{2}$. Denote this probability by d_{j-1}^A.
- The probability that the service provider is a type B agent with $j-1$ many tokens is $(\frac{\beta}{1+\beta})^2((z_{j-1}^B)^2 - (z_j^B)^2) + 2(\frac{1}{1+\beta})(\frac{\beta}{1+\beta})(z_{j-1}^B - z_j^B)z_j^A + 2(\frac{1}{1+\beta})(\frac{\beta}{1+\beta})(z_{j-1}^A - z_j^A)(z_{j-1}^B - z_j^B)\frac{1}{2}$. Denote this probability by d_{j-1}^B.

Hence, we have $\beta_{e_{ij}^A}(\vec{z}) = c_i^A d_{j-1}^A, \beta_{e_{ij}^B}(\vec{z}) = c_i^B d_{j-1}^B, \beta_{(e_i,-e_j)}(\vec{z}) = c_j^B d_{i-1}^A$ and $\beta_{(-e_i,e_j)}(\vec{z}) = c_i^A d_{j-1}^B$. Clearly the condition (5) is satisfied since the jump rate is bounded in the system, and given the constants α and β, the Lipschitz condition can be easily checked. Using Kurtz's theorem, the differential equations, which characterizes the infinite system with asymmetric agents can be found as

$$\frac{dz_i^A}{dt} = -c_i^A + d_{i-1}^A \text{ for all } i \in \mathbb{Z}, \tag{40}$$

$$\frac{dz_i^B}{dt} = -c_i^B + d_{i-1}^B \text{ for all } i \in \mathbb{Z}. \tag{41}$$

M/f	2	4	6	8
1	0.6486	0.6476	0.6475	0.6469
2	0.8787	0.8753	0.8734	0.8706
3	0.9523	0.9510	0.9515	0.9506
4	0.9769	0.9767	0.9777	0.9780

Table 1 The values for $p_{A,f,M}$ where $p_B = 10p_A$ and $q_B = 10q_A$.

M/f	2	4	6	8
1	0.6434	0.6410	0.6398	0.6405
2	0.8684	0.8645	0.8609	0.8587
3	0.9521	0.9512	0.9502	0.9495
4	0.9803	0.9809	0.9816	0.9824

Table 2 The values for $p_{B,f,M}$ where $p_B = 10p_A$ and $q_B = 10q_A$.

Since agents start with 0 tokens and exchange one token at each transition of the system, the expected number of tokens agents have is 0, and it can be translated as follows:

$$\sum_{i\geq 1}^{\infty} z_i^A - \sum_{i\leq 0}^{\infty}(1-z_i^A) + \sum_{i\geq 1}^{\infty} z_i^B - \sum_{i\leq 0}^{\infty}(1-z_i^B) = 0. \tag{42}$$

As we mentioned earlier, we are unable to find a closed form expression for the equilibrium point, but instead one may perform numerical studies using (40), (41) and (42).

We ran simulations in order to conduct comparative statics on having relatively more agents of one type and on the dominance of one type over the other. In both simulations, we fix the number of agents to $n = 10$, and let the system run until $t = 2\cdot 10^7$. Let f be the number of type A agents and $p_{A,f,M}$ be the probability of in the long run, we have $-M \leq s_i^t \leq M$, where $M \in \mathbb{Z}^+$ and i is a type A agent. Similarly define $p_{B,f,M}$. In the first simulation, we fix α and β, and vary f (see Tables 1 and 2). In the second simulation, we fix f and vary p_A (see Tables 3 and 4). Observe that in all simulations, having two types of agents does not create any instability as agents' number of tokens remain between -4 and 4, with high probability.

M/p_A	0.02	0.04	0.06	0.08	0.1
1	0.6478	0.6469	0.6463	0.6453	0.6447
2	0.8752	0.8742	0.8735	0.8726	0.8714
3	0.9518	0.9520	0.9522	0.9527	0.9527
4	0.9774	0.9779	0.9786	0.9794	0.9797

Table 3 The values for $p_{A,5,M}$.

M/p_A	0.02	0.04	0.06	0.08	0.1
1	0.6410	0.6418	0.6431	0.6437	0.6452
2	0.8632	0.8657	0.8681	0.8697	0.8713
3	0.9508	0.9516	0.9524	0.9525	0.9527
4	0.9811	0.9810	0.9807	0.9801	0.9798

Table 4 The values for $p_{B,5,M}$.