# 18.022 2014, Midterm Exam 1

There are 10 problems, and each is worth 10 points. Please explain all your work. You have 50 minutes. Allowed materials are the lecture notes only.

If you solved and understood the homework problems then this should be easy. Good luck!

Name: ______________________

Signature: ______________________

Student ID: ______________________

Recitation leader: ______________________

Recitation number and time: ______________________

| Question | Score |
|---|---|
| 1a | |
| 1b | |
| 2a | |
| 2b | |
| 2c | |
| 2d | |
| 2e | |
| 3a | |
| 3b | |
| 3c | |
| Total | |

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(1) Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, with $\vec{v}$ and $\vec{w}$ not parallel. Let $\Pi_1$ be the plane given parametrically by the equation $\vec{u} + s\vec{v} + t\vec{w}$. Let $\Pi_2$ be the plane given parametrically by the equation $s(\vec{v} + \vec{w}) + t(\vec{v} - \vec{w})$.

   (a) Show that $\Pi_1$ and $\Pi_2$ are parallel.

   (b) Assume now that $|\vec{v}| = |\vec{w}| = 1$ and that $\vec{v}$ and $\vec{w}$ are orthogonal. Calculate the distance between $\Pi_1$ and $\Pi_2$ and give it a geometric interpretation.

(2) A function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is called *idempotent* if $f(f(p)) = f(p)$ for all $p \in \mathbb{R}^n$, and is called an *involution* if $f(f(p)) = p$ for all $p \in \mathbb{R}^n$. Note that you are not expected to have previously encountered these terms; the definitions given in the previous sentence suffice to solve the problem.

   (a) Show that the identity function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = x$ is an idempotent involution. Explain why it is the only function from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is an idempotent involution.

   (b) Find all the linear transformations $f \colon \mathbb{R} \to \mathbb{R}$ that are idempotent. Find all the linear transformations $f \colon \mathbb{R} \to \mathbb{R}$ that are involutions.

   (c) Let $\vec{v} \in \mathbb{R}^n$ be non-zero, and let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(\vec{w}) = \text{proj}_{\vec{v}}\vec{w}$. Show that $f$ is a linear transformation, and that it is idempotent.

   (d) Find three different idempotent linear transformations from $\mathbb{R}^4$ to $\mathbb{R}^4$.

   (e) Let $\vec{u} = (u_1, u_2, u_3)$ have magnitude 1, and let $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(\vec{w}) = \text{proj}_{\vec{u}}\vec{w}$. Write the matrix associated with $f$; this is the matrix $A$ such that $f(\vec{w}) = A\vec{w}$.

(3) Let $f \colon \mathbb{R}^2 \to [0, 2\pi)$ be the function that assigns to each $(x, y) \in \mathbb{R}^2$ the unique angle $0 \leq \theta < 2\pi$ between the positive $x$-axis and the line segment connecting $(0, 0)$ with $(x, y)$. It assigns 0 to $(0, 0)$. This is simply the function that calculates the angle $\theta$ of the polar coordinates on the plane.

   (a) Show that $f$ is a surjection but not an injection.

   (b) What are the $c$-level sets of $f$?

   (c) Show that $f$ is not continuous at $(0, 0)$ by showing that for $\epsilon = \pi/2$ and every $\delta > 0$ there is a $q \in B_\delta(0, 0)$ such that $|f(q) - f(0, 0)| > \epsilon$.