18.022 2014, Midterm Exam 2

Please explain all your work. You have 50 minutes. Allowed materials are the lecture notes only.

Name: ______________________________

Signature: ______________________________

Student ID: ______________________________

Recitation leader: ______________________________

Recitation number and time: ______________________________

Question	Score
1a	
1b	
2a	
2b	
2c	
2d	
2e	
2f	
3a	
3b	
Total	

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(1) (22 points) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^{-x}$, and suppose that $\vec{a} \in \mathbb{R}^3$ is constant. Define $T : \mathbb{R}^3 \to \mathbb{R}$ by

$$T(\vec{x}) = f(|\vec{x} - \vec{a}|^2) = e^{-|\vec{x}-\vec{a}|^2}.$$

We may think of $T(\vec{x})$ as the temperature at $\vec{x} \in \mathbb{R}^3$ resulting from a heat source at $\vec{a}$; note that the temperature only depends on the distance from $\vec{x}$ to $\vec{a}$.

(a) Calculate $\nabla T(\vec{x})$ at a point $\vec{x} \in \mathbb{R}^3$.

(b) Recall that if $\hat{n}$ is a unit vector, then the directional derivative of T in the direction $\hat{n}$ at $\vec{x} \in \mathbb{R}^3$ is given by

$$D_{\hat{n}}T(\vec{x}) = \lim_{h \to 0} \frac{T(\vec{x} + h\hat{n}) - T(\vec{x})}{h}.$$

Fix a point $\vec{x} \in \mathbb{R}^3$. For which unit vectors $\hat{n}$ is $D_{\hat{n}}T(\vec{x})$ equal to zero? (i.e., in which directions does the temperature not change?). Give a short geometric interpretation of your answer.

(2) (56 points) Suppose $\vec{x}\colon \mathbb{R} \to \mathbb{R}^2$ is a regular curve of class $\mathcal{C}^2$, parametrized by arclength (so that $|\vec{x}'(t)| = 1$). Let $\lambda > 0$ and $\theta \in \mathbb{R}$ be constants, let

$$M = \begin{pmatrix} \lambda\cos\theta & -\lambda\sin\theta \\ \lambda\sin\theta & \lambda\cos\theta \end{pmatrix},$$

and let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(\vec{x}) = M\vec{x}$. Let $\vec{r}\colon \mathbb{R} \to \mathbb{R}^2$ be given by $\vec{r}(t) = g(\vec{x}(t))$.

(a) Let $\vec{v} \in \mathbb{R}^2$ be a non-zero vector. Show that $|g(\vec{v})|$, the norm of $g(\vec{v})$, is equal to $\lambda|\vec{v}|$, and that the angle between $\vec{v}$ and $g(\vec{v})$ is θ.

(b) Calculate the partial derivatives of g ($\frac{\partial g_i}{\partial x_j}$ for $i = 1,2$ and $j = 1,2$), and explain why g is differentiable. What is the matrix Dg (that is, the derivative of g)?

(c) Calculate $\vec{r}'(t)$ in terms of $\vec{x}'(t)$. (Hint: use the chain rule.)

(d) Calculate $s_r(t)$, the arclength parametrization of $\vec{r}$, with respect to the reference point $t = 0$ (this is the distance along the curve $\vec{r}$ from 0 to t). (Hint: Recall that $\vec{x}(t)$ is parametrized by arclength. Calculate $s_r'(t)$ and then integrate.)

(e) Calculate $\vec{T}_r(t)$, the unit tangent vector of $\vec{r}$, in terms of $\vec{T}_x(t)$, the unit tangent vector of $\vec{x}$.

(f) **Extra credit:** Calculate $\vec{N}_r(t)$, the unit normal vector of $\vec{r}$ as well as $\kappa_r(t)$, the curvature of $\vec{r}$, in terms of $\vec{N}_x(t)$ and $\kappa_x(t)$, the unit normal vector of $\vec{x}$ and the curvature of $\vec{x}$.

(3) (22 points) Given an $f\colon \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^2$, recall that the *Hessian Matrix* $H = (h_{ij})$ is the n-by-n matrix given by

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Suppose $M = (m_{ij})$ is a symmetric 2×2 matrix (i.e., $m_{ij} = m_{ji}$), suppose $\vec{a} \in \mathbb{R}^2$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\vec{x}) = \vec{a} \cdot \vec{x} + \tfrac{1}{2}\vec{x} \cdot (M\vec{x}).$$

This can also be written as

$$f(x_1, x_2) = a_1x_1 + a_2x_2 + \tfrac{1}{2}(x_1m_{11}x_1 + x_1m_{12}x_2 + x_2m_{21}x_1 + x_2m_{22}x_2).$$

Note that (a_i) and (m_{ij}) are constants.

(a) Show that the gradient of f at $\vec{x} \in \mathbb{R}^2$ is equal to $\vec{a}+M\vec{x}$, and calculate f's Hessian.

(b) Assume now that $\det M \neq 0$, so that M is invertible. Solve the equation $\nabla f(p) = \vec{0}$, and calculate the tangent plane to the graph of f at the solution p, in terms of $f(p)$.