18.022 2014, Midterm Exam 3

Please explain all your work. You have 50 minutes. Allowed materials are the lecture notes only.

Each question is worth 11 points, except the bonus question which is worth 5, and 1(e) which is worth 12.

Name: ______________________

Signature: ______________________

Student ID: ______________________

Recitation leader: ______________________

Recitation number and time: ______________________

Question	Score
1a	
1b	
1c	
1d	
1e	
2a	
2b	
2c	
2d	
2e	
Total	

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(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function which attains a local minimum at $a \in \mathbb{R}$, and such that $f(a) \neq 0$ and $f''(a) > 0$. (e.g., $f(x) = (x-a)^2 + 1$ satisfies all of these conditions).

Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = f(x)f(y)$. Express your answers in terms of f and its derivatives.

(a) Calculate ∇g, and show that $(a,a) \in \mathbb{R}^2$ is a critical point of g.

(b) Calculate the Hessian of g.

(c) Is (a, a) a local minimum or a local maximum (or neither) of g? Hint: this depends on $f(a)$.

(d) Calculate $P_{(a,a),1}g$, the first order Taylor polynomial of g, centered at (a, a).

(e) Assume now that $f(x) = e^{(x-a)^2}$, so that $g(x, y) = e^{(x-a)^2+(y-a)^2}$. Calculate $P_{(a,a),2}g$, the second order Taylor polynomial of g, centered at (a, a).

(2) Let $W = \{(x, y, z) \in \mathbb{R}^3 \, : \, x^2 + y^2 \leq 2, -1 \leq z \leq 1 - x^2 - y^2\}$ be the subset of $\mathbb{R}^3$ bounded between the plane $z = -1$ and the graph of the paraboloid $g(x, y) = 1 - x^2 - y^2$.

Hint: use cylindrical coordinates.

(a) Draw a simple diagram of W, describe its shape and calculate its volume.

(b) We would like to find a point on the boundary of W that is closest to the origin. We will do this by finding a minimum on the boundary of W of the squared distance function $f(x, y, z) = x^2 + y^2 + z^2$.

Explain which properties of the boundary of W and f guarantee that such a point exists.

(c) Use Lagrange multipliers to find critical points of f restricted to the boundary of W.
Since we are looking for minima, there is no need to look on the plane $z = -1$: any point there is at least as far as to the origin as $(0, 0, 1)$. Hence it is enough to consider the "top half" of the boundary of W, $\{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 \leq 2, z = 1 - x^2 - y^2\}$.

(d) Find a point on the boundary of W that is closest to the origin.

(e) **Bonus question.** Integrate the function $g(x, y, z) = \cosh(x+y)\sinh(x-y)e^z$ over W.