18.022 2014, 4^{TH} MIDTERM (TAKE-HOME), DUE FRIDAY, DECEMBER 5^{TH}

This assignment is extra credit, and will count as an additional midterm if doing so will raise the final grade. Please work on this on your own. Allowed materials are this course's lecture notes, problems sets and their solutions, and the course book.

Each part is worth 10 points.

(1) let $D \subset \mathbb{R}^2$ be the region enclosed by the curve $r = g(\theta)$, for some $\mathcal{C}^1$, non-negative $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x + 2\pi) = g(x)$ for all $x \in \mathbb{R}$.

(a) Calculate the length of ∂D, the boundary of D. Express your answer as an integral involving g and its first derivative.

(b) Let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D. Calculate the length of ∂D again, but this time express the answer as an integral involving the derivatives of x and y.

(c) Calculate the length of ∂D for the case that $g(\theta) = 1 - \cos\theta$.

(d) In the remainder of this problem we will prove an important theorem, called the *isoperimetric inequality* (this proof is due to E. Schmidt, from 1938): it states that the length of the boundary of any shape on the plane is at least equal to the square root of 4π times its area.

Let C be the unit circle. Explain why

$$\text{area}(D) + \pi = \oint_{\partial D} (0, x) \cdot \mathrm{d}\vec{s} + \oint_C (-y, 0) \cdot \mathrm{d}\vec{s}.$$

(e) Assume henceforth that $g(x) \leq 1$ and $g(0) = g(\pi) = 1$. Show that

$$(x(\theta), w(\theta)) = \begin{cases} (g(\theta)\cos\theta, \sqrt{1 - g(\theta)^2\cos^2\theta} & 0 \leq \theta \leq \pi \\ (g(\theta)\cos\theta, -\sqrt{1 - g(\theta)^2\cos^2\theta} & \pi \leq \theta \leq 2\pi \end{cases}$$

is a parametrization of a unit circle.

(f) Let $(x(\theta), w(\theta))$ be the parametrization of the unit circle C from 1e. Again let

$$(x(\theta), y(\theta)) = (g(\theta)\cos\theta, g(\theta)\sin\theta)$$

be a parametrization of ∂D. Using 1d, show that

$$\text{area}(D) + \pi = \int_0^{2\pi} (x(\theta), -w(\theta)) \cdot (y'(\theta), x'(\theta))\, \mathrm{d}\theta.$$

(g) Explain why it follows from the previous question that

$$\text{area}(D) + \pi \leq \oint_0^{2\pi} \sqrt{(x(\theta)^2 + w(\theta)^2)(x'(\theta)^2 + y'(\theta)^2)}\, \mathrm{d}\theta.$$

(h) Explain why

$$\text{area}(D) + \pi \leq \text{length}(\partial D).$$

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(i) Recall the geometric-arithmetic mean inequality: for $a, b > 0$ it holds that $\sqrt{ab} \le (a+b)/2$. Use this to show that

$$\sqrt{4\pi \cdot \mathrm{area}(D)} \le \mathrm{length}(\partial D).$$

For which shape are these two quantities equal?

(2) In this question we will prove the Divergence Theorem for a special class of vector fields, and for cubic regions.

Let $\vec{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field, given as the sum

$$\vec{F}(x, y, z) = u(x)\hat{\imath} + v(y)\hat{\jmath} + w(z)\hat{k},$$

for some $\mathcal{C}^1$ functions $u, v, w \colon \mathbb{R} \to \mathbb{R}$, so that the first component of $\vec{F}(x, y, z)$ depends on x, the second depends only on y, and the third depends only on z.

Given $h > 0$, let C_h be the cube whose vertices are $\vec{0}, h\hat{\imath}, h\hat{\jmath}, h\hat{k}, h(\hat{\imath} + \hat{\jmath}), h(\hat{\imath} + \hat{k}), h(\hat{k} + \hat{\jmath})$ and $h(\hat{\imath} + \hat{\jmath} + \hat{k})$.

(a) Calculate

$$\iint_{\partial C_h} \vec{F} \cdot d\vec{S}$$

in terms of u, v, w and h. Hint: make sure to orient the normals to point out of C_h.

(b) Calculate

$$\iiint_{C_h} \operatorname{div} \vec{F} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

the integral of the divergence of $\vec{F}$ over C_h, in terms of u, v, w and h, and conclude that

$$\iiint_{C_h} \operatorname{div} \vec{F} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial C_h} \vec{F} \cdot d\vec{S},$$

which is (a special case of) the Divergence Theorem.