Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Based on lecture notes by James McKernan

0.1. Definition.

Blackboard 1. *A* **vector** $\vec{u} \in \mathbb{R}^3$ *is a 3-tuple of real numbers* (v_1, v_2, v_3).

- *Examples:* $(2014, -1, 17.3)$, $(-1, \sqrt{2}, \pi)$, $\vec{0} = (0, 0, 0)$.
- *Order matters:* $(1, 2, 3) \neq (1, 3, 2)$.
- *Can also be written in a column:* $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

Vectors in $\mathbb{R}^2$ have two numbers, but otherwise are the same.

0.2. Addition and multiplication by scalars.

Blackboard 2. *Let* $\vec{v}$ *and* $\vec{w}$ *be two vectors in* $\mathbb{R}^3$. *Then their* **sum** *is*

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}.$$

Example: If $\vec{v} = (2, -3, 1)$ *and* $\vec{w} = (1, -5, 3)$ *then* $\vec{v} + \vec{w} = (3, -8, 4)$.

Subtraction can be defined similarly.

Geometric / graphical interpretation: think of vectors as displacements from the origin. Then to build $\vec{v} + \vec{w}$, shift $\vec{w}$ so that its tail is connected to $\vec{v}$'s head. The vector connecting the tail of $\vec{v}$ and the head of the shifted $\vec{w}$ is $\vec{v} + \vec{w}$. Subtraction is done by connecting the heads without shifting: $\vec{w} + (\vec{v} - \vec{w}) = \vec{v}$.

Blackboard 3. *The* **product** *of* $\vec{v} \in \mathbb{R}^3$ *and* $\lambda \in \mathbb{R}$ *(called a* **scalar***) is*

$$\lambda \vec{v} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}.$$

Example: If $\vec{v} = (2, -3, 1)$, $\lambda = -3$ *then* $\lambda \vec{v} = (-6, 9, -3)$.

Theorem 4. *If* λ, μ *are scalars and* $\vec{v}, \vec{w}, \vec{u}$ *are vectors then*

(1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
(2) $\vec{v} + \vec{0} = \vec{v}$.
(3) $\vec{v} + (\vec{w} + \vec{u}) = (\vec{v} + \vec{w}) + \vec{u}$.
(4) $\lambda(\vec{v} + \vec{w}) = \lambda \vec{v} + \lambda \vec{w}$.
(5) $(\lambda + \mu)\vec{v} = \lambda \vec{v} + \mu \vec{v}$.
(6) $\lambda(\mu \vec{v}) = (\lambda \mu)\vec{v}$.

Proof. Proof of (1):

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \\ w_3 + v_3 \end{pmatrix} = \vec{w} + \vec{v}.$$

□

0.3. **Magnitudes.**

Blackboard 5. *The* **magnitude** *(or* **length** *or* **norm***) of* $\vec{v} \in \mathbb{R}^3$ *is*

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^3}.$$

The **direction** *of a vector* $\vec{v} \in \mathbb{R}^3$ *is the vector*

$$\hat{v} = \frac{1}{|\vec{v}|}\vec{v}.$$

A vector with magnitude one is called a **unit vector***.*

Theorem 6. *The magnitude of* $\lambda\vec{v}$ *is* $|\lambda|$ *times the magnitude of* $\vec{v}$*.*

Proof.

$$\begin{aligned} |\lambda\vec{v}| &= |(\lambda v_1, \lambda v_2, \lambda v_3)| \\ &= \sqrt{(\lambda v_1)^2 + (\lambda v_2)^2 + (\lambda v_3)^3} \\ &= \lambda\sqrt{v_1^2 + v_2^2 + v_3^3} \\ &= |\lambda||\vec{v}|. \end{aligned}$$

□

Corollary 7. *The magnitude of* $\frac{1}{|\vec{v}|}\vec{v}$ *is one.*

Proof. Substitute $\lambda = \frac{1}{|\vec{v}|}$ in the theorem. □

Geometric / graphical interpretation:

- Multiplying by $\lambda > 0$ changes the length by λ but leaves the direction the same (proof?).
- Multiplying by $\lambda = 1$ does nothing.
- Multiplying by $\lambda = 0$ always results in $\vec{0}$.
- Multiplying by $\lambda < 0$ changes the length by $|\lambda|$ and changes the direction to the opposite direction.
- Multiplying by $\lambda = -1$ gives the same vector, in the opposite direction.

0.4. **The standard basis.**

Blackboard 8. $\mathbb{R}^3$ *has three special unit vectors called the* **standard basis**

$$\hat{e}_1 = \hat{\imath} = (1, 0, 0) \quad \hat{e}_2 = \hat{\jmath} = (0, 1, 0) \quad \hat{e}_3 = \hat{k} = (0, 0, 1).$$

Vector $\vec{v} = (v_1, v_2, v_3)$ *can be written as the* **linear combination** $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$.

0.5. **Physics.** Vectors are important in math, and also in physics, engineering etc. For example

Blackboard 9.

- *Velocity, acceleration, force and electric field are vectors.*
- *Energy and mass are not vectors.*
- $\vec{F} = m\vec{a}$ *is the vector form of Newton's equation.*
- *Acceleration due to gravity is*

$$\vec{a} = -\frac{Gm_1m_2}{|\vec{r}|^3}\vec{r}.$$

0.6. **Parametrizing lines using vectors.**

Blackboard 10. *Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}^3$.*

Characterize all the vectors $\vec{c}$ whose heads lie on the line connecting the heads of $\vec{v}$ and $\vec{w}$.

Let $\vec{a} = \vec{v} - \vec{w}$. Then $\vec{c} = \vec{w} + t\vec{a}$ for some $0 \le t \le 1$.

We can also write this as $\vec{c} = \vec{w} + t\vec{a} = \vec{w} + t(\vec{v} - \vec{w}) = (1-t)\vec{w} + t\vec{v}$.

The entire line that the heads lie on is $\vec{c} = \vec{w} + t\vec{a}$, where t can be any number.

Blackboard 11. *Let $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3)$ be points in space. Then $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$ is the vector from P to Q.*

Example: Let $P = (1, 2)$, $Q = (4, -5)$. The line PQ can be parametrized by

$$P + t(\overrightarrow{PQ}) = (1, 2) + t(3, -7) = (1 + 3t, 2 - 7t).$$

Where do the two lines $(1, -2 + 2s, 3 - 4s)$ and $(-1 + 2t, -3 + t, 3t)$ intersect?

- *By the first component $1 = -1 + 2t$ so $t = 1$.*
- *By the second component $-2 + 2s = -3 + 1$ so $s = 0$.*
- *We have equality in the third component. Otherwise the lines wouldn't intersect.*
- *So the lines intersect at $(1, -2, 3)$.*

Theorem 12. *Medians of a triangle intersect at the same point, and the intersection point divides them in proportion $2 : 1$.*

Triangle ABC, D between A and C, E between C and B, F between B and A, G the intersection of DB and AE. Let $\vec{a} = \overrightarrow{CA}$ and $\vec{b} = \overrightarrow{CB}$.

Proof. Points of AE are $\vec{a} + t(\frac{1}{2}\vec{b} - \vec{a}) = (1-t)\vec{a} + \frac{1}{2}t\vec{b}$.

Points of BD are $\vec{b} + s(\frac{1}{2}\vec{a} - \vec{b}) = (1-s)\vec{b} + \frac{1}{2}s\vec{a}$.

Intersection at $\frac{1}{2}s = 1 - t$ and $\frac{1}{2}t = 1 - s$. Hence $t = \frac{2}{3}$ and

$$\overrightarrow{CG} = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b}.$$

Now, $\overrightarrow{EG} + \overrightarrow{GC} = \overrightarrow{EC}$. Therefore

$$\begin{aligned} \overrightarrow{EG} &= \overrightarrow{CG} - \tfrac{1}{2}\vec{b} \\ &= \tfrac{1}{3}\vec{a} + \tfrac{1}{3}\vec{b} - \tfrac{1}{2}\vec{b} \\ &= \tfrac{1}{3}(\vec{a} - \tfrac{1}{2}\vec{b}) \\ &= \tfrac{1}{3}\overrightarrow{EA}. \end{aligned}$$

By symmetry, the third median intersects at the same point. □