More about derivatives
Based on lecture notes by James McKernan

The main result is:

Theorem 1. *Let $S \subset \mathbb{R}^n$ be an open subset and let $f\colon S \to \mathbb{R}^m$ be a function. If the partial derivatives*

$$\frac{\partial f_i}{\partial x_j},$$

exist and are continuous, then f is differentiable.

We will need:

Theorem 2 (Mean value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ is continuous and differentiable at every point of (a,b), then we may find $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

Geometrically, (2) is clear. However it is surprisingly hard to give a complete proof.

Proof of (1). We may assume that $m = 1$. We only prove this in the case when $n = 2$ (the general case is similar, only notationally more involved). So we have

$$f\colon \mathbb{R}^2 \to \mathbb{R}.$$

Suppose that $p = (a,b)$ and let $\overrightarrow{pq} = h_1\hat{\imath} + h_2\hat{\jmath}$. Let

$$p_0 = (a,b) \qquad p_1 = (a+h_1, b) \qquad \text{and} \qquad p_2 = (a+h_1, b+h_2) = q.$$

Now

$$f(q) - f(p) = [f(p_2) - f(p_1)] + [f(p_1) - f(p_0)].$$

We apply the Mean value theorem twice. We may find q_1 and q_2 such that

$$f(p_1) - f(p_0) = \frac{\partial f}{\partial x}(q_1)h_1 \qquad \text{and} \qquad f(p_2) - f(p_1) = \frac{\partial f}{\partial y}(q_2)h_2.$$

Here q_1 lies somewhere on the line segment p_0p_1 and q_2 lies on the line segment p_1p_2. Putting this together, we get

$$f(q) - f(p) = \frac{\partial f}{\partial x}(q_1)h_1 + \frac{\partial f}{\partial y}(q_2)h_2.$$

Thus

$$\begin{aligned}
\frac{|f(q) - f(p) - A\cdot\overrightarrow{pq}|}{|\overrightarrow{pq}|} &= \frac{|(\frac{\partial f}{\partial x}(q_1) - \frac{\partial f}{\partial x}(p))h_1 + (\frac{\partial f}{\partial y}(q_2) - \frac{\partial f}{\partial y}(p))h_2|}{|\overrightarrow{pq}|} \\
&\leq \frac{|(\frac{\partial f}{\partial x}(q_1) - \frac{\partial f}{\partial x}(p))h_1|}{|\overrightarrow{pq}|} + \frac{|(\frac{\partial f}{\partial y}(q_2) - \frac{\partial f}{\partial y}(p))h_2|}{|\overrightarrow{pq}|} \\
&\leq \frac{|(\frac{\partial f}{\partial x}(q_1) - \frac{\partial f}{\partial x}(p))h_1|}{|h_1|} + \frac{|(\frac{\partial f}{\partial y}(q_2) - \frac{\partial f}{\partial y}(p))h_2|}{|h_2|} \\
&= |(\frac{\partial f}{\partial x}(q_1) - \frac{\partial f}{\partial x}(p))| + |(\frac{\partial f}{\partial y}(q_2) - \frac{\partial f}{\partial y}(p))|.
\end{aligned}$$

Note that as q approaches p, q_1 and q_2 both approach p as well. As the partials of f are continuous, we have

$$\lim_{q\to p}\frac{|f(q)-f(p)-A\cdot\overrightarrow{pq}|}{|\overrightarrow{pq}|}\leq \lim_{q\to p}(|(\frac{\partial f}{\partial x}(q_1)-\frac{\partial f}{\partial x}(p))|+|(\frac{\partial f}{\partial y}(q_2)-\frac{\partial f}{\partial y}(p))|)=0.$$

Therefore f is differentiable at p, with derivative A. □

Example 3. *Let $f\colon S\to\mathbb{R}$ be given by*

$$f(x,y)=\frac{x}{\sqrt{x^2+y^2}},$$

where $S=\mathbb{R}^2-\{(0,0)\}$. Then

$$\frac{\partial f}{\partial x}=\frac{(x^2+y^2)^{1/2}-x(2x)(1/2)(x^2+y^2)^{-1/2}}{x^2+y^2}=\frac{y^2}{(x^2+y^2)^{3/2}}.$$

Similarly

$$\frac{\partial f}{\partial y}=-\frac{xy}{(x^2+y^2)^{3/2}}.$$

Now both partial derivatives exist and are continuous, and so f is differentiable, with derivative the gradient,

$$\nabla f=(\frac{y^2}{(x^2+y^2)^{3/2}},-\frac{xy}{(x^2+y^2)^{3/2}})=\frac{1}{(x^2+y^2)^{3/2}}(y^2,-xy).$$

Lemma 4. *Let $A=(a_{ij})$ be an $m\times n$ matrix. Let*

$$z=\sqrt{\sum_{i,j}a_{ij}^2}.$$

If $\vec{v}\in\mathbb{R}^n$ then

$$|A\vec{v}|\leq z|\vec{v}|.$$

Proof. Let $a_1,\dots,a_m$ be the rows of A. Then the entry in the ith row of $A\vec{v}$ is $\vec{a}_i\cdot\vec{v}$. So,

$$\begin{aligned}|A\vec{v}|^2&=(\vec{a}_1\cdot\vec{v})^2+(\vec{a}_2\cdot\vec{v})^2+\cdots+(\vec{a}_n\cdot\vec{v})^2\\&\leq|\vec{a}_1|^2|\vec{v}|^2+|\vec{a}_2|^2|\vec{v}|^2+\cdots+|\vec{a}_n|^2|\vec{v}|^2\\&=(|\vec{a}_1|^2+|\vec{a}_2|^2+\cdots+|\vec{a}_n|^2)|\vec{v}|^2\\&=z^2|\vec{v}|^2.\end{aligned}$$

Now take square roots of both sides. □

Theorem 5. *Let $f\colon S\to\mathbb{R}^m$ be a function, where $S\subset\mathbb{R}^n$ is open.*

If f is differentiable at p, then f is continuous at p.

Proof. Suppose that $Df(p)=A$. Then

$$\lim_{q\to p}\frac{f(q)-f(p)-A\cdot\overrightarrow{pq}}{|\overrightarrow{pq}|}=0.$$

This is the same as to require

$$\lim_{q\to p}\frac{|f(q)-f(p)-A\cdot\overrightarrow{pq}|}{|\overrightarrow{pq}|}=0.$$

But if this happens, then surely

$$\lim_{q \to p} |f(q) - f(p) - A \cdot \overrightarrow{pq}| = 0.$$

So

$$\begin{aligned} |f(q) - f(p)| &= |f(q) - f(p) - A \cdot \overrightarrow{pq} + A \cdot \overrightarrow{pq}| \\ &\leq |f(q) - f(p) - A \cdot \overrightarrow{pq}| + |A \cdot \overrightarrow{pq}| \\ &\leq |f(q) - f(p) - A \cdot \overrightarrow{pq}| + z|\overrightarrow{pq}|. \end{aligned}$$

Taking the limit as q approaches p, both terms on the RHS go to zero, so that

$$\lim_{q \to p} |f(q) - f(p)| = 0,$$

and f is continuous at p. □