Higher derivatives
Based on lecture notes by James McKernan

We first record a very useful fact:

Theorem 1. *Let $A \subset \mathbb{R}^n$ be an open subset. Let $f\colon A \to \mathbb{R}^m$ and $g\colon A \to \mathbb{R}^m$ be two functions and suppose that $p \in A$. Let $\lambda \in A$ be a scalar.*

If f and g are differentiable at p, then

1. *$f + g$ is differentiable at p and $D(f+g)(p) = Df(p) + Dg(p)$.*
2. *$\lambda \cdot f$ is differentiable at p and $D(\lambda f)(p) = \lambda D(f)(p)$.*

Now suppose that $m = 1$.

3. *fg is differentiable at p and $D(fg)(p) = D(f)(p)g(p) + f(p)D(g)(p)$.*
4. *If $g(p) \neq 0$, then fg is differentiable at p and*

$$D(f/g)(p) = \frac{D(f)(p)g(p) - f(p)D(g)(p)}{g^2(p)}.$$

If the partial derivatives of f and g exist and are continuous, then (1) follows from the well-known single variable case. One can prove the general case of (1), by hand (basically lots of ϵ's and δ's). However, perhaps the best way to prove (1) is to use the chain rule, proved in the next section.

What about higher derivatives?

Blackboard 2. *Let $A \subset \mathbb{R}^n$ be an open set and let $f\colon A \to \mathbb{R}$ be a function. The k**th order partial derivative** of f, **with respect to the variables** x_{i_1}, x_{i_2}, $\dots x_{i_k}$ is the iterated derivative*

$$\frac{\partial^k f}{\partial x_{i_k}\partial x_{i_{k-1}}\dots\partial x_{i_2}\partial x_{i_1}}(p) = \frac{\partial}{\partial x_{i_k}}(\frac{\partial}{\partial x_{i_{k-1}}}(\dots\frac{\partial}{\partial x_{i_2}}(\frac{\partial f}{\partial x_{i_1}})\dots))(p).$$

We will also use the notation $f_{x_{i_k}x_{i_{k-1}}\dots x_{i_2}x_{i_1}}(p)$.

Example 3. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,t) = e^{-at}\cos x$.*

Then

$$\begin{aligned} f_{xx}(x,t) &= \frac{\partial}{\partial x}(\frac{\partial}{\partial x}(e^{-at}\cos x)) \\ &= \frac{\partial}{\partial x}(-e^{-at}\sin x) \\ &= -e^{-at}\cos x. \end{aligned}$$

On the other hand,

$$\begin{aligned} f_{xt}(x,t) &= \frac{\partial}{\partial x}(\frac{\partial}{\partial t}(e^{-at}\cos x)) \\ &= \frac{\partial}{\partial x}(-ae^{-at}\cos x) \\ &= ae^{-at}\sin x. \end{aligned}$$

Similarly,

$$\begin{aligned} f_{tx}(x,t) &= \frac{\partial}{\partial t}(\frac{\partial}{\partial x}(e^{-at}\cos x)) \\ &= \frac{\partial}{\partial t}(-e^{-at}\sin x) \\ &= ae^{-at}\sin x. \end{aligned}$$

Note that

$$f_t(x,t) = -ae^{-at}\cos x.$$

It follows that $f(x,t)$ *is a solution to the* ***Heat equation:***

$$a\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}.$$

Blackboard 4. *Let* $A \subset \mathbb{R}^n$ *be an open subset and let* $f\colon A \to \mathbb{R}^m$ *be a function. We say that* f *is of* ***class*** $\mathcal{C}^k$ *if all* k*th partial derivatives exist and are continuous.*

We say that f *is of* ***class*** $\mathcal{C}^\infty$ *(aka* ***smooth****) if* f *is of class* $\mathcal{C}^k$ *for all* k*.*

In lecture 10 we saw that if f is $\mathcal{C}^1$, then it is differentiable.

Theorem 5. *Let* $A \subset \mathbb{R}^n$ *be an open subset and let* $f\colon A \to \mathbb{R}^m$ *be a function.*

If f *is* $\mathcal{C}^2$*, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i},$$

for all $1 \leq i, j \leq n$*.*

The proof uses the Mean Value Theorem.

Suppose we are given $A \subset \mathbb{R}$ an open subset and a function $f\colon A \to \mathbb{R}$ of class $\mathcal{C}^1$. The objective is to find a solution to the equation

$$f(x) = 0.$$

Newton's method proceeds as follows. Start with some $x_0 \in A$. The best linear approximation to $f(x)$ in a neighbourhood of x_0 is given by

$$f(x_0) + f'(x_0)(x - x_0).$$

If $f'(x_0) \neq 0$, then the linear equation

$$f(x_0) + f'(x_0)(x - x_0) = 0,$$

has the unique solution,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now just keep going (assuming that $f'(x_i)$ is never zero),

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ \vdots &= \vdots \\ x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \end{aligned}$$

Claim 6. *Suppose that* $x_\infty = \lim_{n\to\infty} x_n$ *exists and* $f'(x_\infty) \neq 0$*.*

Then $f(x_\infty) = 0$*.*

Proof of (6). Indeed, we have

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Take the limit as n goes to ∞ of both sides:

$$x_\infty = x_\infty - \frac{f(x_\infty)}{f'(x_\infty)},$$

we used the fact that f and f' are continuous and $f'(x_\infty) \neq 0$. But then

$$f(x_\infty) = 0,$$

as claimed. □

Suppose that $A \subset \mathbb{R}^n$ is open and $f \colon A \to \mathbb{R}^n$ is a function. Suppose that f is $\mathcal{C}^1$ (that is, suppose each of the coordinate functions $f_1, \dots, f_n$ is $\mathcal{C}^1$).

The objective is to find a solution to the equation

$$f(p) = \vec{0}.$$

Before we do this, we'll need to define determinants and inverses of matrices.

Blackboard 7. *The identity n-by-n matrix I_n has 1's on the diagonal and zeros elsewhere. Let A be an n-by-n matrix.*

Claim: $IA = AI = A$.

An n-by-n matrix B is an "inverse of A" if $AB = BA = I$. A is "invertible" if it has an inverse.

Blackboard 8. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The determinant of A, $\det A$, is $ad - bc$.

Claim 9. *If $\det A \neq 0$ then*

$$B = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

is the unique inverse of A.

Blackboard 10. *One can also define determinants for $n \times n$ matrices. It is probably easiest to explain the general rule using an example:*

$$\begin{vmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix}.$$

Notice that we as expand about the top row, the sign alternates $+-+-$, so that the last term comes with a minus sign.

Claim 11. *Let A be an n-by-n matrix. If $\det A \neq 0$ then A has a unique inverse.*

Back to solving $f(p) = \vec{0}$. Start with any point $p_0 \in A$. The best linear approximation to f at p_0 is given by

$$f(p_0) + Df(p_0)\overrightarrow{pp_0}.$$

Assume that $Df(p_0)$ is an invertible matrix, that is, assume that $\det Df(p_0) \neq 0$. Then the inverse matrix $Df(p_0)^{-1}$ exists and the unique solution to the linear equation

$$f(p_0) + Df(p_0)\overrightarrow{pp_0} = \vec{0},$$

is given by

$$p_1 = p_0 - Df(p_0)^{-1} f(p_0).$$

Notice that matrix multiplication is not commutative, so that there is a difference between $Df(p_0)^{-1} f(p_0)$ and $f(p_0) Df(p_0)^{-1}$. If possible, we get a sequence of solutions,

$$\begin{aligned} p_1 &= p_0 - Df(p_0)^{-1} f(p_0) \\ p_2 &= p_1 - Df(p_1)^{-1} f(p_1) \\ \vdots &= \vdots \\ p_n &= p_{n-1} - Df(p_{n-1})^{-1} f(p_{n-1}). \end{aligned}$$

Suppose that the limit $p_\infty = \lim_{n\to\infty} p_n$ exists and that $Df(p_\infty)$ is invertible. As before, if we take the limit of both sides, this implies that

$$f(p_\infty) = \vec{0}.$$

Let us try a concrete example.

Example 12. *Solve*

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= x^3. \end{aligned}$$

First we write down an appropriate function, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $f(x, y) = (x^2 + y^2 - 1, y^2 - x^3)$. Then we are looking for a point p such that

$$f(p) = (0, 0).$$

Then

$$Df(p) = \begin{pmatrix} 2x & 2y \\ -3x^2 & 2y \end{pmatrix}.$$

The determinant of this matrix is

$$4xy + 6x^2 y = 2xy(2 + 3x).$$

Now if we are given a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we may write down the inverse by hand,

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So

$$Df(p)^{-1} = \frac{1}{2xy(2 + 3x)} \begin{pmatrix} 2y & -2y \\ 3x^2 & 2x \end{pmatrix}$$

So,

$$\begin{aligned} Df(p)^{-1} f(p) &= \frac{1}{2xy(2 + 3x)} \begin{pmatrix} 2y & -2y \\ 3x^2 & 2x \end{pmatrix} \begin{pmatrix} x^2 + y^2 - 1 \\ y^2 - x^3 \end{pmatrix} \\ &= \frac{1}{2xy(2 + 3x)} \begin{pmatrix} 2x^2 y - 2y + 2x^3 y \\ x^4 + 3x^2 y^2 - 3x^2 + 2xy^2 \end{pmatrix} \end{aligned}$$

One nice thing about this method is that it is quite easy to implement on a computer. Here is what happens if we start with $(x_0, y_0) = (5, 2)$,

$$
\begin{aligned}
(x_0, y_0) &= (5.00000000000000, 2.00000000000000)\\
(x_1, y_1) &= (3.24705882352941, -0.617647058823529)\\
(x_2, y_2) &= (2.09875150983980, 1.37996311951634)\\
(x_3, y_3) &= (1.37227480405610, 0.561220968705054)\\
(x_4, y_4) &= (0.959201654346683, 0.503839504009063)\\
(x_5, y_5) &= (0.787655203525685, 0.657830227357845)\\
(x_6, y_6) &= (0.755918792660404, 0.655438554539110),
\end{aligned}
$$

and if we start with $(x_0, y_0) = (5, 5)$,

$$
\begin{aligned}
(x_0, y_0) &= (5.00000000000000, 5.00000000000000)\\
(x_1, y_1) &= (3.24705882352941, 1.85294117647059)\\
(x_2, y_2) &= (2.09875150983980, 0.363541705259258)\\
(x_3, y_3) &= (1.37227480405610, -0.306989760884339)\\
(x_4, y_4) &= (0.959201654346683, -0.561589294711320)\\
(x_5, y_5) &= (0.787655203525685, -0.644964218428458)\\
(x_6, y_6) &= (0.755918792660404, -0.655519172668858).
\end{aligned}
$$

One can sketch the two curves and check that these give reasonable solutions. One can also check that (x_6, y_6) lie close to the two given curves, by computing $x_6^2 + y_6^2 - 1$ and $y_6^2 - x_6^3$.