Curvature and torsion

Based on lecture notes by James McKernan

Blackboard 1. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^2$ regular curve (i.e., $\vec{r}'(t) \neq \vec{0}$ for all t).*

The ***curvature*** *$\kappa(s)$ of $\vec{r}(s)$ is the magnitude of the vector*

$$\vec{T}'(s) = \frac{d\vec{T}(s)}{ds},$$

and the ***unit normal vector*** *$\vec{N}$ is the unit vector pointing in the direction of $\vec{T}'(s)$*

$$\vec{N}(s) = \frac{\vec{T}'(s)}{\|\vec{T}'(s)\|}.$$

One can try to calculate the curvature using the parameter t. By the chain rule,

$$\frac{d\vec{T}(t)}{dt} = \frac{d\vec{T}(s)}{ds}\frac{ds}{dt}.$$

So

$$\frac{d\vec{T}(s)}{ds} = \frac{\frac{d\vec{T}(t)}{dt}}{\frac{ds}{dt}}.$$

The denominator is the speed. It follows that $\vec{n}$ and $d\vec{T}/dt$ point in the same direction.

Note that the normal vector and the unit tangent vector are always orthogonal. Indeed, more generally

Proposition 2. *Let $\vec{v}\colon I \to \mathbb{R}^n$. Then*

$$\frac{d(\vec{v}\cdot\vec{v})}{dt} = 2\vec{v}'\cdot\vec{v},$$

and in particular if $|\vec{v}(t)|$ is constant then $\vec{v}'$ and $\vec{v}$ are orthogonal.

Now,

$$\|\vec{T}(s)\| = 1.$$

and so, as $\vec{N}(s)$ points in the same direction as $\vec{T}'(s)$, it follows that the tangent vector and the normal vector are orthogonal.

Blackboard 3.

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s).$$

is called the ***binormal vector****.*

The three vectors $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ are unit vectors and pairwise orthogonal, that is, these vectors are an orthonormal basis of $\mathbb{R}^3$. Notice that $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ are a right handed set.

We call these vectors a **moving frame** or the **Frenet-Serret frame**. Now

$$\frac{d\vec{B}}{ds}(s)\cdot\vec{B}(s) = 0,$$

as

$$\vec{B}(s)\cdot\vec{B}(s) = 1.$$

It follows that

$$\frac{d\vec{B}}{ds}(s),$$

lies in the plane spanned by $\vec{T}(s)$ and $\vec{N}(s)$.

$$\begin{aligned}
\frac{d\vec{B}}{ds}(s)\cdot\vec{T}(s) &= \frac{d(\vec{T}\times\vec{N})}{ds}(s)\cdot\vec{T}(s)\\
&= \left(\frac{d\vec{T}}{ds}(s)\times\vec{N}(s)+\vec{T}(s)\times\frac{d\vec{N}}{ds}(s)\right)\cdot\vec{T}(s)\\
&= \kappa(s)(\vec{N}(s)\times\vec{N}(s))\cdot\vec{T}(s)+(\vec{T}(s)\times\frac{d\vec{N}}{ds}(s))\cdot\vec{T}(s)\\
&= 0+(\vec{T}(s)\times\vec{T}(s))\cdot\frac{d\vec{N}}{ds}(s)\\
&= 0.
\end{aligned}$$

It follows that

$$\frac{d\vec{B}}{ds}(s) \qquad \text{and} \qquad \vec{T}(s),$$

are orthogonal, and so

$$\frac{d\vec{B}}{ds}(s) \qquad \text{is parallel to} \qquad \vec{N}(s).$$

Blackboard 4. *The **torsion** of the curve $\vec{r}(s)$ is the unique scalar $\tau(s)$ such that*

$$\frac{d\vec{B}}{ds}(s) = -\tau(s)\vec{N}(s).$$

If we have a helix, the sign of the torsion distinguishes between a right handed helix and a left handed helix. The magnitude of the torsion measures how spread out the helix is (the curvature measures how tight the turns are). Now

$$\frac{d\vec{N}}{ds}(s)$$

is orthogonal to $\vec{N}(s)$, and so it is a linear combination of $\vec{T}(s)$ and $\vec{B}(s)$. In fact,

$$\begin{aligned}
\frac{d\vec{N}}{ds}(s) &= \frac{d(\vec{B}\times\vec{T})}{ds}(s)\\
&= \frac{d\vec{B}}{ds}(s)\times\vec{T}(s)+\vec{B}(s)\times\frac{d\vec{T}}{ds}(s)\\
&= -\tau(s)\vec{N}(s)\times\vec{T}(s)+\kappa(s)\vec{B}(s)\times\vec{N}(s)\\
&= \tau(s)\vec{B}(s)-\kappa(s)\vec{T}(s)\\
&= -\kappa(s)\vec{T}(s)+\tau(s)\vec{B}(s).
\end{aligned}$$

Blackboard 5. *We say that $\vec{r}(t)$ is **smooth** if $\vec{r}(t)$ is $\mathcal{C}^\infty$.*

Theorem 6 (Frenet Formulae)**.** *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth parametrised curve. Then*

$$\begin{pmatrix}\vec{T}'(s)\\ \vec{N}'(s)\\ \vec{B}'(s)\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{pmatrix}.$$

Of course, s represents the arclength parameter and primes denote derivatives with respect to s. Notice that the 3×3 matrix A appearing in (6) is **skew-symmetric**, that is $A^t = -A$. The way we have written the Frenet formulae, it appears that we have two 3×1 vectors; strictly speaking these are the rows of two 3×3 matrices.

Theorem 7. *Let $I \subset \mathbb{R}$ be an open interval and suppose we are given two smooth functions*

$$\kappa\colon I \longrightarrow \mathbb{R} \qquad \text{and} \qquad \tau\colon I \longrightarrow \mathbb{R},$$

where $\kappa(s) > 0$ for all $s \in I$.

Then there is a regular smooth curve $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ parametrised by arclength with curvature $\kappa(s)$ and torsion $\tau(s)$. Further, any two such curves are congruent, that is, they are the same up to translation and rotation.

Let's consider the example of the helix:

Example 8.

$$\vec{r}(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c}),$$

where

$$c^2 = a^2 + b^2.$$

Let's assume that $a > 0$. By convention $c > 0$. Then

$$\vec{T}(s) = \frac{1}{c}(-a \sin \frac{s}{c}, a \cos \frac{s}{c}, b).$$

Hence

$$\frac{dT}{ds}(s) = \frac{-a}{c^2}(\cos \frac{s}{c}, \sin \frac{s}{c}, 0) = \frac{a}{c^2}(-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0) = \frac{a}{c^2}\vec{N}(s)$$

It follows that

$$\kappa(s) = \frac{a}{c^2} \qquad \text{and} \qquad \vec{N}(s) = (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0).$$

Finally,

$$\vec{B}(s) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{bs}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{vmatrix}$$

It follows that

$$\vec{B}(s) = (\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c}) = \frac{1}{c}(b \sin \frac{s}{c}, -b \cos \frac{s}{c}, a).$$

Finally, note that

$$\frac{d\vec{B}}{ds}(s) = \frac{b}{c^2}(\cos \frac{s}{c}, \sin \frac{s}{c}, 0) = -\frac{b}{c^2}\vec{N}.$$

Using this we can compute the torsion:

$$\tau(s) = \frac{b}{c^2}.$$

It is interesting to use the torsion and curvature to characterise various geometric properties of curves. Let's say that a parametrised differentiable curve $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ is **planar** if there is a plane Π which contains the image of $\vec{r}$.

Theorem 9. *A regular smooth curve $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ is planar if and only if the torsion is zero.*

Proof. We may assume that the curve passes through the origin.

Suppose that $\vec{r}$ is planar. Then the image of $\vec{r}$ is contained in a plane Π. As the curve passes through the origin, Π contains the origin as well. Note that the unit tangent vector $\vec{T}(s)$ and the unit normal vector $\vec{N}(s)$ are contained in Π. It follows that $\vec{B}(s)$ is a normal vector to the plane; as $\vec{B}(s)$ is a unit vector, it must be constant. But then

$$\frac{d\vec{B}}{ds}(s) = \vec{0} = 0\vec{N}(s),$$

so that the torsion is zero.

Now suppose that the torsion is zero. Then

$$\frac{dB}{ds}(s) = 0\vec{N} = \vec{0},$$

so that $\vec{B}(s) = B_0$, is a constant vector. Consider the function

$$f(s) = \vec{r}(s) \cdot \vec{B}(s) = \vec{r}(s) \cdot \vec{B}_0.$$

Then

$$\begin{aligned} \frac{df}{ds}(s) &= \frac{d(\vec{r} \cdot \vec{B}_0)}{ds}(s) \\ &= \vec{T}(s) \cdot \vec{B}_0 = 0. \end{aligned}$$

So $f(s)$ is constant. It is zero when $\vec{r}(a) = \vec{0}$ (the curve passes through the origin) so that $f(s) = 0$. But then $\vec{r}(s)$ is always orthogonal to a fixed vector, so that $\vec{r}$ is contained in a plane, that is, C is planar. □

It is interesting to try to figure out how to characterise curves which are contained in spheres or cylinders.