Vector fields
Based on lecture notes by James McKernan

Blackboard 1. *Let $A \subset \mathbb{R}^n$ be an open subset. A* ***vector field*** *on A is function $\vec{F}\colon A \longrightarrow \mathbb{R}^n$.*

One obvious way to get a vector field is to take the gradient of a differentiable function. If $f\colon A \longrightarrow \mathbb{R}$, then

$$\nabla f\colon A \longrightarrow \mathbb{R}^n,$$

is a vector field.

Blackboard 2. *A vector field $\vec{F}\colon A \longrightarrow \mathbb{R}^n$ is called a* ***gradient (aka conservative) vector field*** *if $\vec{F} = \nabla f$ for some differentiable function $f\colon A \longrightarrow \mathbb{R}$.*

Example 3. *Let*

$$\vec{F}\colon \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}^3,$$

be the vector field

$$\vec{F}(x,y,z) = \frac{cx}{(x^2+y^2+z^2)^{3/2}}\hat{\imath} + \frac{cy}{(x^2+y^2+z^2)^{3/2}}\hat{\jmath} + \frac{cz}{(x^2+y^2+z^2)^{3/2}}\hat{k},$$

for some constant c. Then $\vec{F}(x,y,z)$ is the gradient of

$$f\colon \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R},$$

given by

$$f(x,y,z) = -\frac{c}{(x^2+y^2+z^2)^{1/2}}.$$

So $\vec{F}$ is a conservative vector field. Notice that if $c<0$ then $\vec{F}$ models the gravitational force and f is the potential (note that unfortunately mathematicians and physicists have different sign conventions for f).

Proposition 4. *If $\vec{F}$ is a conservative vector field and $\vec{F}$ is $\mathcal{C}^1$ function, then*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

for all i and j between 1 and n.

Proof. If $\vec{F}$ is conservative, then we may find a differentiable function $f\colon A \longrightarrow \mathbb{R}^n$ such that

$$F_i = \frac{\partial f}{\partial x_i}.$$

As F_i is $\mathcal{C}^1$ for each i, it follows that f is $\mathcal{C}^2$. But then

$$\begin{aligned} \frac{\partial F_i}{\partial x_j} &= \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= \frac{\partial^2 f}{\partial x_i \partial x_j} \\ &= \frac{\partial F_j}{\partial x_i}. \end{aligned}$$

□

Notice that (4) is a negative result; one can use it show that various vector fields are not conservative.

Example 5. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{F}(x,y) = (-y,x).$$

Then

$$\frac{\partial F_1}{\partial y} = -1 \qquad \textit{and} \qquad \frac{\partial F_2}{\partial x} = 1 \neq -1.$$

So $\vec{F}$ is not conservative.

Example 6. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{F}(x,y) = (y,x+y).$$

Then

$$\frac{\partial F_1}{\partial y} = 1 \qquad \textit{and} \qquad \frac{\partial F_2}{\partial x} = 1,$$

so $\vec{F}$ might be conservative. Let's try to find

$$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \textit{such that} \qquad \nabla f(x,y) = (y,x+y).$$

If f exists, then we must have

$$\frac{\partial f}{\partial x} = y \qquad \textit{and} \qquad \frac{\partial f}{\partial y} = x+y.$$

If we integrate the first equation with respect to x, then we get

$$f(x,y) = xy + g(y).$$

Note that $g(y)$ is not just a constant but it is a function of y. There are two ways to see this. One way, is to imagine that for every value of y, we have a separate differential equation. If we integrate both sides, we get an arbitrary constant c. As we vary y, c varies, so that $c = g(y)$ is a function of y. On the other hand, if to take the partial derivatives of $g(y)$ with respect to x, then we get 0. Now we take $xy + g(y)$ and differentiate with respect to y, to get

$$x + y = \frac{\partial(xy + g(y))}{\partial y} = x + \frac{dg}{dy}(y).$$

So

$$g'(y) = y.$$

Integrating both sides with respect to y we get

$$g(y) = y^2/2 + c.$$

It follows that

$$\nabla(xy + y^2/2) = (y, x+y),$$

so that $\vec{F}$ is conservative.