Flow lines
Based on lecture notes by James McKernan

Blackboard 1. *If $\vec{F}\colon A \longrightarrow \mathbb{R}^n$ is a vector field, we say that a parametrised differentiable curve $\vec{r}\colon I \longrightarrow A$ is a **flow line** for $\vec{F}$, if*

$$\vec{r}'(t) = \vec{F}(\vec{r}(t)),$$

for all $t \in I$.

Note: if $r(t)$ is a flow line for $\vec{F}$ then

$$\vec{T}(t) = \frac{\vec{F}(\vec{r}(t))}{|\vec{F}(\vec{r}(t))|}.$$

Example 2. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{F}(x,y) = (-y,x).$$

We check that

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{r}(t) = (a\cos t, a\sin t),$$

is a flow line. In fact

$$\vec{r}'(t) = (-a\sin t, a\cos t),$$

and so

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= \vec{F}(a\cos t, a\sin t)\\ &= \vec{r}'(t),\end{aligned}$$

so that $\vec{r}(t)$ is indeed a flow line.

Example 3. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{F}(x_1,x_2) = (-x_1,x_2).$$

Let's find a flow line $\vec{r}(t)$ through the point (a,b). We have

$$(r_1'(t), r_2'(t)) = F(r_1(t), r_2(t)) = (-r_1(t), r_2(t)).$$

If we denote $x(t) = r_1(t)$ and $y(t) = r_2(t)$ then

$$\begin{aligned}x'(t) &= -x(t) & \qquad x(0) &= a\\ y'(t) &= y(t) & \qquad y(0) &= b.\end{aligned}$$

Therefore,

$$x(t) = ae^{-t} \qquad \textit{and} \qquad y(t) = be^t,$$

gives the flow line through (a,b).

Example 4. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{F}(x,y) = (x^2-y^2, 2xy).$$

Try

$$\begin{aligned}x(t) &= 2a\cos t\sin t\\ y(t) &= 2a\sin^2 t.\end{aligned}$$

Then

$$\begin{aligned} x'(t) &= 2a(-\sin^2 t + \cos^t) \\ &= \frac{x^2(t) - y^2(t)}{y(t)}. \end{aligned}$$

Similarly

$$\begin{aligned} y'(t) &= 4a \cos t \sin t \\ &= \frac{2x(t)y(t)}{y(t)}. \end{aligned}$$

So

$$\vec{r}'(t) = \frac{\vec{F}(\vec{r}(t))}{f(t)}.$$

So this isn't a flow line. But

$$\vec{T}(t) = \frac{\frac{\vec{F}(\vec{r}(t))}{f(t)}}{|\frac{\vec{F}(\vec{r}(t))}{f(t)}|} = \frac{\vec{F}(\vec{r}(t))}{|\vec{F}(\vec{r}(t))|},$$

and so the directions are correct. Hence the curves themselves are flow lines, but this is not the correct parametrisation. The flow lines are circles passing through the origin, with centre along the y-axis.

Example 5. *Let*

$$\vec{F} \colon \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x,y) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}).$$

Then

$$\frac{\partial F_1}{\partial y}(x,y) = -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

and

$$\frac{\partial F_2}{\partial x}(x,y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

So $\vec{F}$ might be conservative. Let's find the flow lines. Try

$$\begin{aligned} x(t) &= a \cos\left(\frac{t}{a^2}\right) \\ y(t) &= a \sin\left(\frac{t}{a^2}\right). \end{aligned}$$

Then

$$\begin{aligned} x'(t) &= -\frac{1}{a} \sin\left(\frac{t}{a^2}\right) \\ &= -\frac{y}{x^2+y^2}. \end{aligned}$$

Similarly

$$\begin{aligned} y'(t) &= \frac{1}{a} \cos\left(\frac{t}{a^2}\right) \\ &= \frac{x}{x^2+y^2}. \end{aligned}$$

So the flow lines are closed curves. In fact this means that $\vec{F}$ is not conservative.

Theorem 6. *Let $A \subset \mathbb{R}^n$ be open and let $T\colon A \longrightarrow \mathbb{R}$ be a differentiable function.*

If $\vec{r}\colon I \longrightarrow A$ is a flow line for $\nabla T\colon A \longrightarrow \mathbb{R}^n$, then the function $T \circ \vec{r}\colon I \longrightarrow \mathbb{R}$ is increasing.

Proof. By the chain rule,

$$\begin{aligned} \frac{d(T \circ \vec{r})}{dt}(t) &= \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) \\ &= \vec{r}'(t) \cdot \vec{r}'(t) \geq 0. \end{aligned}$$

□

Corollary 7. *A closed parametrised curve is never the flow line of a conservative vector field.*

Once again, note that (7) is mainly a negative result:

Example 8.

$$\vec{F}\colon \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 \qquad \textit{given by} \qquad \vec{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

is not a conservative vector field as it has flow lines which are circles.