MAXIMA AND MINIMA: II

BASED ON LECTURE NOTES BY JAMES MCKERNAN

To see how to maximize and minimize a function on the boundary, let's conside a concrete example.

Let

$$K = \{ (x, y) \mid x^2 + y^2 \leq 2 \}.$$

Then K is compact. Let

$$f \colon K \longrightarrow \mathbb{R},$$

be the function $f(x, y) = xy$. Then f is continuous and so f achieves its maximum and minimum.

I. Let's first consider the interior points. Then

$$\nabla f(x, y) = (y, x),$$

so that $(0, 0)$ is the only critical point. The Hessian of f is

$$Hf(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$d_1 = 0$ and $d_2 = -1 \neq 0$ so that $(0, 0)$ is a saddle point.

It follows that the maxima and minima of f are on the boundary, that is, the set of points

$$C = \{ (x, y) \mid x^2 + y^2 = 2 \}.$$

II. Let $g \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $g(x, y) = x^2 + y^2$. Then the circle C is a level curve of g. The original problem asks to maximize and minimize

$$f(x, y) = xy \qquad \text{subject to} \qquad g(x, y) = x^2 + y^2 = 2.$$

One way to proceed is to use the second equation to eliminate a variable. The method of Lagrange multipliers does exactly the opposite. Instead of eliminating a variable we add one more variable, traditionally called λ.

In general, say we want to maximize $f(x, y)$ subject to $g(x, y) = c$. Then at a maximum point p it won't necessarily be the case that $\nabla f(p) = 0$, but it will be the case that the *directional derivative* $\nabla f(p) \cdot \hat{n}$ will be zero for any $\hat{n}$ that is in the direction of the level set $g(x, y) = c$. Since ∇g is orthogonal to this level set, at a maximum point p it will be the case that $\nabla f(p)$ and $\nabla g(p)$ will be at the same direction, or that $\nabla f(p) = \lambda \nabla g(p)$ for some λ.

Consider the function

$$h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c).$$

Let's see what happens at points where $\nabla h = 0$. Taking the derivatives with respect to x and y and equating to zero yields

$$\nabla f(x, y) - \lambda \nabla g(x, y) = 0,$$

which is what we're looking for. Taking the derivative with respect to λ and equating to zero yields

$$g(x, y) = c,$$

which is the second condition we need. Hence finding a point in which $\nabla h = 0$ is the same as solving our problem.

So now let's maximize and minimize

$$h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 2) = xy - \lambda(x^2 + y^2 - 2).$$

We find the critical points of $h(x, y, \lambda)$:

$$\begin{aligned} y &= 2\lambda x \\ x &= 2\lambda y \\ 2 &= x^2 + y^2. \end{aligned}$$

First note that if $x = 0$ then $y = 0$ and $x^2 + y^2 = 0 \neq 2$, impossible. So $x \neq 0$. Similarly one can check that $y \neq 0$ and $\lambda \neq 0$. Divide the first equation by the second:

$$\frac{y}{x} = \frac{x}{y},$$

so that $y^2 = x^2$. As $x^2 + y^2 = 2$ it follows that $x^2 = y^2 = 1$. So $x = \pm 1$ and $y = \pm 1$. This gives four potential points $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$. Then the maximum value of f is 1, and this occurs at the first and the last point. The minimum value of f is -1, and this occurs at the second and the third point.

One can also try to parametrize the boundary:

$$\vec{r}(t) = \sqrt{2}(\cos t, \sin t).$$

So we maximize the composition

$$h \colon [0, 2\pi] \longrightarrow \mathbb{R},$$

where $h(t) = 2 \cos t \sin t$. As $I = [0, 2\pi]$ is compact, h has a maximum and minimum on I. When $h'(t) = 0$, we get

$$\cos^2 t - \sin^2 t = 0.$$

Note that the LHS is $\cos 2t$, so we want

$$\cos 2t = 0.$$

It follows that $2t = \pi/2 + 2m\pi$, so that

$$t = \pi/4, \qquad 3\pi/4, \qquad 5\pi/4, \qquad \text{and} \qquad 7\pi/4.$$

These give the four points we had before.

What is the closest point to the origin on the surface

$$F = \{\, (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, xyz = p \,\}?$$

So we want to minimize the distance to the origin on F. The first trick is to minimize the square of the distance. In other words, we are trying to minimize $f(x, y, z) = x^2 + y^2 + z^2$ on the surface

$$F = \{\, (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, xyz = p \,\}.$$

In words, given three numbers $x \geq$, $y \geq 0$ and $z \geq 0$ whose product is $p > 0$, what is the minimum value of $x^2 + y^2 + z^2$?

Now F is closed but it is not bounded, so it is not even clear that the minimum exists.

Let's use the method of Lagrange multipliers. Let

$$h \colon \mathbb{R}^4 \longrightarrow \mathbb{R},$$

be the function

$$h(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xyz - p).$$

We look for the critical points of h:

$$\begin{aligned} 2x &= \lambda yz \\ 2y &= \lambda xz \\ 2z &= \lambda xy \\ p &= xyz. \end{aligned}$$

Once again, it is not possible for any of the variables to be zero. Taking the product of the first three equations, we get

$$8(xyz) = \lambda^3(x^2y^2z^2).$$

So, dividing by xyz and using the last equation, we get

$$8 = \lambda^3 p,$$

that is

$$\lambda = \frac{2}{p^{1/3}}.$$

Taking the product of the first two equations, and dividing by xy, we get

$$4 = \lambda^2 z^2,$$

so that

$$z = p^{1/3}.$$

So $h(x, y, z, \lambda)$ has a critical point at

$$(x, y, z, \lambda) = (p^{1/3}, p^{1/3}, p^{1/3}, \frac{2}{p^{1/3}}).$$

We check that the point

$$(x, y, z) = (p^{1/3}, p^{1/3}, p^{1/3}),$$

is a minimum of $x^2 + y^2 + z^2$ subject to the constraint $xyz = p$. At this point the sum of the squares is

$$3p^{2/3}.$$

Suppose that $x \geq 2p^{1/3}$. Then the sum of the squares is at least $4p^{2/3}$. Similarly if $y \geq 2p^{1/3}$ or $z \geq 2p^{1/3}$. On the other hand, the set

$$K = \{\, (x, y, z) \in \mathbb{R}^3 \mid x \in [0, 2p^{1/3}], y \in [0, 2p^{1/3}], z \in [0, 2p^{1/3}], xyz = p \,\},$$

is closed and bounded, so that f achieves it minimum on this set, which we have already decided is at

$$(x, y, z) = (p^{1/3}, p^{1/3}, p^{1/3}),$$

since f is larger on the boundary. Putting all of this together, the point

$$(x, y, z) = (p^{1/3}, p^{1/3}, p^{1/3}),$$

is a point where the sum of the squares is a minimum.

Here is another such problem. Find the closest point to the origin which also belongs to the cone

$$x^2 + y^2 = z^2,$$

and to the plane

$$x + y + z = 3.$$

As before, we minimize $f(x, y, z) = x^2+y^2+z^2$ subject to $g_1(x, y, z) = x^2+y^2-z^2 = 0$ and $g_2(x, y, z) = x+y+z = 3$. Introduce a new function, with two new variables λ_1 and λ_2,

$$h \colon \mathbb{R}^5 \longrightarrow \mathbb{R},$$

given by

$$\begin{aligned} h(x, y, z, \lambda_1, \lambda_2) &= f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z) \\ &= x^2 + y^2 + z^2 - \lambda_1(x^2 + y^2 - z^2) - \lambda_2(x + y + z - 3). \end{aligned}$$

We find the critical points of h:

$$\begin{aligned} 2x &= 2\lambda_1 x + \lambda_2 \\ 2y &= 2\lambda_1 y + \lambda_2 \\ 2z &= -2\lambda_1 z + \lambda_2 \\ z^2 &= x^2 + y^2 \\ 3 &= x + y + z. \end{aligned}$$

Suppose we substract the first equation from the second:

$$y - x = \lambda_1(y - x).$$

So either $x = y$ or $\lambda_1 = 1$. Suppose $x \neq y$. Then $\lambda_1 = 1$ and $\lambda_2 = 0$. In this case $z = -z$, so that $z = 0$. But then $x^2 + y^2 = 0$ and so $x = y = 0$, which is not possible.

It follows that $x = y$, in which case $z = \pm\sqrt{2}x$ and

$$(2 \pm \sqrt{2})x = 3.$$

So

$$x = \frac{3}{2 \pm \sqrt{2}} = \frac{3(2 \mp \sqrt{2})}{2}.$$

This gives us two critical points:

$$\begin{aligned} p &= (\frac{3(2 - \sqrt{2})}{2}, \frac{3(2 - \sqrt{2})}{2}, \frac{3\sqrt{2}(2 - \sqrt{2})}{2}) \\ q &= (\frac{3(2 + \sqrt{2})}{2}, \frac{3(2 + \sqrt{2})}{2}, -\frac{3\sqrt{2}(2 - \sqrt{2})}{2}). \end{aligned}$$

Of the two, clearly the first is closest to the origin.

To finish, we had better show that this point is the closest to the origin on the whole locus

$$F = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, x + y + z = 3 \}.$$

Let

$$K = \{ (x, y, z) \in F \mid x^2 + y^2 + z^2 \leq 25 \}.$$

Then K is closed and bounded, whence compact. So f achieves its minimum somewhere on K, and so it must achieve its minimum at p. Clearly outside f is at least 25 on $F \setminus K$, and so f is a minimum at p on the whole of F.