More double integrals
Based on lecture notes by James McKernan

Proposition 1. *Let $D = D_1 \cup D_2$ be a bounded region and let $f\colon D \longrightarrow \mathbb{R}$ be a function.*

If f is integrable over D_1 and over D_2, then f is integrable over D and and $D_1 \cap D_2$, and we have

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_1 \cap D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Example 2. *Let*

$$D = \{\, (x,y) \in \mathbb{R}^2 \,|\, 1 \leq x^2 + y^2 \leq 9 \,\}.$$

Then D is not an elementary region. Let

$$D_1 = \{\, (x,y) \in D \,|\, y \geq 0 \,\} \qquad \textit{and} \qquad D_2 = \{\, (x,y) \in D \,|\, y \leq 0 \,\}.$$

Then D_1 and D_2 are both of type 1.

If f is continuous, then f is integrable over D and $D_1 \cap D_2$. In fact

$$\begin{aligned} D_1 \cap D_2 = L \cup R = \{\, (x,y) \in \mathbb{R}^2 \,|\, -3 \leq x \leq -1, 0 \leq y \leq 0 \,\} \\ \cup \{\, (x,y) \in \mathbb{R}^2 \,|\, 1 \leq x \leq 3, 0 \leq y \leq 0 \,\}. \end{aligned}$$

Now L and R are elementary regions. We have

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^3 \left(\int_0^0 f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = 0.$$

Therefore, by symmetry,

$$\iint_L f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

and so

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

To integrate f over D_1, break D_1 into three parts.

$$\begin{aligned} \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-3}^{3} \left(\int_{\gamma(x)}^{\delta(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_{-3}^{-1} \left(\int_0^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x \\ &+ \int_{-1}^{1} \left(\int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x \\ &+ \int_{1}^{3} \left(\int_0^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x. \end{aligned}$$

One can do something similar for D_2.

Example 3. *Calculate the volume of a solid ball of radius a. Let*

$$B = \{\, (x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq a^2 \,\}.$$

We want the volume of B. Break into two pieces. Let

$$B^+ = \{\, (x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq a^2, z \geq 0 \,\}.$$

Let

$$D = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq a^2 \,\}.$$

Then B^+ *is bounded by the xy-plane and the graph of the function*

$$f \colon D \longrightarrow \mathbb{R},$$

given by

$$f(x, y) = \sqrt{a^2 - x^2 - y^2}.$$

It follows that

$$\begin{aligned}
\mathrm{vol}(B^+) &= \iint_D \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{1 - \frac{y^2}{a^2 - x^2}}\,\sqrt{a^2 - x^2}\,\mathrm{d}y \right) \mathrm{d}x.
\end{aligned}$$

Now let's make the substitution

$$t = \frac{y}{\sqrt{a^2 - x^2}} \qquad \textit{so that} \qquad dt = \frac{dy}{\sqrt{a^2 - x^2}}.$$

$$\begin{aligned}
\mathrm{vol}(B^+) &= \int_{-a}^{a} \left(\int_{-1}^{1} \sqrt{1 - t^2}(a^2 - x^2)\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_{-a}^{a} (a^2 - x^2) \left(\int_{-1}^{1} \sqrt{1 - t^2}\,\mathrm{d}t \right) \mathrm{d}x
\end{aligned}$$

Now let's make the substitution

$$t = \sin u \qquad \textit{so that} \qquad dt = \cos u\,du.$$

$$\begin{aligned}
\mathrm{vol}(B^+) &= \int_{-a}^{a} (a^2 - x^2) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u \right) \mathrm{d}x \\
&= \int_{-a}^{a} (a^2 - x^2) \frac{\pi}{2}\,\mathrm{d}x \\
&= \frac{\pi}{2} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a} \\
&= \pi(a^3 - \frac{a^3}{3}) \\
&= \frac{2\pi a^3}{3}.
\end{aligned}$$

Therefore, we get the expected answer

$$\mathrm{vol}(B) = 2\,\mathrm{vol}(B^+) = \frac{4\pi a^3}{3}.$$

Example 4. *Now consider the example of a cone whose base radius is a and whose height is b. Put the central axis along the x-axis and the base in the yz-plane. In the xy-plane we get an equilateral triangle of height b and base $2a$. If we view this as a region of type 1, we have*

$$\gamma(x) = -a\left(1 - \frac{x}{b}\right) \qquad \text{and} \qquad \delta(x) = a\left(1 - \frac{x}{b}\right).$$

We want to integrate the function

$$f \colon D \longrightarrow \mathbb{R},$$

given by

$$f(x, y) = \sqrt{a^2\left(1 - \frac{x}{b}\right)^2 - y^2}.$$

So half of the volume of the cone is

$$\begin{aligned}
\int_0^b \left(\int_{-a(1-\frac{x}{b})}^{a(1-\frac{x}{b})} \sqrt{a^2\left(1 - \frac{x}{b}\right)^2 - y^2}\, \mathrm{d}y \right) \mathrm{d}x &= \frac{\pi}{2} \int_0^b a^2 \left(1 - \frac{x}{b}\right)^2 \mathrm{d}x \\
&= \frac{\pi a^2}{2} \int_0^b 1 - \frac{2x}{b} + \frac{x^2}{b^2}\, \mathrm{d}x \\
&= \frac{\pi a^2}{2} \left[x - \frac{x^2}{b} + \frac{x^3}{3b^2} \right]_0^b \\
&= \frac{1}{6}(\pi a^2 b).
\end{aligned}$$

Therefore the volume is

$$\frac{1}{3}(\pi a^2 b).$$