TRIPLE INTEGRALS
BASED ON LECTURE NOTES BY JAMES MCKERNAN

Blackboard 1. *Let $B = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3$ be a box in space. A* ***partition*** *$\mathcal{P}$ of R is a triple of sequences:*

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_n = b \\ c = y_0 < y_1 < \cdots < y_n = d \\ e = z_0 < z_1 < \cdots < z_n = f. \end{aligned}$$

The ***mesh*** *of $\mathcal{P}$ is*

$$m(\mathcal{P}) = \max\{\, x_i - x_{i-1}, y_i - y_{i-1}, z_i - z_{i-1} \,|\, 1 \leq i \leq k \,\}.$$

Now suppose we are given a function

$$f \colon B \longrightarrow \mathbb{R}$$

Pick

$$\vec{c}_{ijk} \in B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_i, z_{i-1}].$$

Blackboard 2. *The sum*

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(\vec{c}_{ijk})(x_i - x_{i-1})(y_j - y_{j-1})(z_i - z_{i-1}),$$

is called a ***Riemann sum****.*

Blackboard 3. *The function $f \colon B \longrightarrow \mathbb{R}$ is called* ***integrable****, with integral I, if for every $\epsilon > 0$, we may find a $\delta > 0$ such that for every mesh $\mathcal{P}$ whose mesh size is less than δ, we have*

$$|I - S| < \epsilon,$$

where S is any Riemann sum associated to $\mathcal{P}$.

If $W \subset \mathbb{R}^3$ is a bounded subset and $f \colon W \longrightarrow \mathbb{R}$ is a bounded function, then pick a box B containing W and extend f by zero to a function $\tilde{f} \colon B \longrightarrow \mathbb{R}$,

$$\tilde{f}(x) = \begin{cases} x & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

If $\tilde{f}$ is integrable, then we write

$$\iiint_W f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_B \tilde{f}(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

In particular

$$\mathrm{vol}(W) = \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

There are two pairs of results, which are much the same as the results for double integrals:

Proposition 4. *Let $W \subset \mathbb{R}^2$ be a bounded subset and let $f \colon W \longrightarrow \mathbb{R}$ and $g \colon W \longrightarrow \mathbb{R}$ be two integrable functions. Let λ be a scalar.*

Then

(1) *$f + g$ is integrable over W and*

$$\iiint_W f(x, y, z) + g(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_W f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_W g(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(2) λf *is integrable over* W *and*

$$\iiint_W \lambda f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \lambda \iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(3) *If* $f(x,y,z) \leq g(x,y,z)$ *for any* $(x,y,z) \in W$*, then*

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq \iiint_W g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(4) $|f|$ *is integrable over* W *and*

$$|\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z| \leq \iiint_W |f(x,y,z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Proposition 5. *Let* $W = W_1 \cup W_2 \subset \mathrm{R}^3$ *be a bounded subset and let* $f\colon W \longrightarrow \mathbb{R}$ *be a bounded function.*

If f *is integrable over* W_1 *and over* W_2*, then* f *is integrable over* W *and and* $W_1 \cap W_2$*, and we have*

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W_1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_{W_2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$- \iiint_{W_1 \cap W_2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Blackboard 6. *Define three maps*

$$\pi_{ij}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2,$$

by projection onto the i*th and* j*th coordinate.*

In coordinates, we have

$$\pi_{12}(x,y,z) = (x,y), \qquad \pi_{23}(x,y,z) = (y,z), \qquad \text{and} \qquad \pi_{13}(x,y,z) = (x,z).$$

For example, if we start with a solid pyramid and project onto the xy-plane, the image is a square, but it project onto the xz-plane, the image is a triangle. Similarly onto the yz-plane.

Blackboard 7. *A bounded subset* $W \subset \mathbb{R}^3$ *is an* ***elementary subset*** *if it is one of four types:*

Type 1: $D = \pi_{12}(W)$ *is an elementary region and*

$$W = \{\, (x,y,z) \in \mathbb{R}^2 \,|\, (x,y) \in D, \epsilon(x,y) \leq z \leq \phi(x,y) \,\},$$

where $\epsilon\colon D \longrightarrow \mathbb{R}$ *and* $\phi\colon D \longrightarrow \mathbb{R}$ *are continuous functions.*

Type 2: $D = \pi_{23}(W)$ *is an elementary region and*

$$W = \{\, (x,y,z) \in \mathbb{R}^2 \,|\, (y,z) \in D, \alpha(y,z) \leq x \leq \beta(y,z) \,\},$$

where $\alpha\colon D \longrightarrow \mathbb{R}$ *and* $\beta\colon D \longrightarrow \mathbb{R}$ *are continuous functions.*

Type 3: $D = \pi_{13}(W)$ *is an elementary region and*

$$W = \{\, (x,y,z) \in \mathbb{R}^2 \,|\, (x,z) \in D, \gamma(x,z) \leq y \leq \delta(x,z) \,\},$$

where $\gamma\colon D \longrightarrow \mathbb{R}$ *and* $\delta\colon D \longrightarrow \mathbb{R}$ *are continuous functions.*

The solid pyramid is of type 4.

Theorem 8. *Let $W \subset \mathbb{R}^3$ be an elementary region and let $f\colon W \longrightarrow \mathbb{R}$ be a continuous function.*

If W is of type 1, then

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\pi_{12}(W)} \left(\int_{\epsilon(x,y)}^{\phi(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y.$$

There are similar statements for types 2 and 3.

Given a type 1 $W \subset \mathbb{R}^3$ and a $z \in \mathbb{R}$, let $W_z = \{(x,y) \in \mathbb{R}^2 \,|\, (x,y,z) \in W\}$. Note that $W_z \subset \mathbb{R}^2$.

Theorem 9. *Let $W \subset \mathbb{R}^3$ be an elementary compact region and let $f\colon W \longrightarrow \mathbb{R}$ be a continuous function. Let $z_{\mathrm{min}} = \min\{z \in \mathbb{R} \,|\, \exists x, y \text{ s.t. } (x,y,z) \in W\}$. Define z_{max} likewise. If W is of type 1, then*

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z_{\mathrm{min}}}^{z_{\mathrm{max}}} \left(\iint_{W_z} f(x,y,z)\,\mathrm{d}x\mathrm{d}y \right) \mathrm{d}z.$$

Again there are similar statements for types 2 and 3.

Let's figure out the volume of the solid ellipsoid:

$$W = \{\, (x,y,z) \in \mathbb{R}^3 \,|\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \,\}.$$

This is an elementary region of type 4.

$$\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{-a}^{a} \left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} \left(\int_{-c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}}^{c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}} \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} 2c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&= 2c\int_{-a}^{a} \left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} \sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \frac{2c}{b}\int_{-a}^{a} \left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} \sqrt{b^2\left(1-\left(\frac{x}{a}\right)^2\right)-y^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \frac{\pi c}{b}\int_{-a}^{a} b^2\left(1-\left(\frac{x}{a}\right)^2\right) \mathrm{d}x \\
&= \pi bc\int_{-a}^{a} 1-\left(\frac{x}{a}\right)^2 \mathrm{d}x
\end{aligned}$$

We will change variables $u = x/a$. Then

$$
\begin{aligned}
&= \pi bc \int_{-a}^{a} \left(1 - u^2\right) a \,\mathrm{d}x \\
&= \pi abc \left[u - u^3/3\right]_{-1}^{1} \\
&= \frac{4\pi}{3} abc.
\end{aligned}
$$

Let's use Theorem 9. $z_{\min} = -c$ and $z_{\max} = +c$. Also,

$$
\begin{aligned}
W_z &= \{(x,y) \,|\, (x/a)^2 + (y/b)^2 \le 1 - (z/c)^2\} \\
&= \left\{ (x,y) \,\middle|\, \left(\frac{x}{a\sqrt{1-(z/c)^2}}\right)^2 + \left(\frac{y}{b\sqrt{1-(z/c)^2}}\right)^2 \le 1 \right\}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathrm{vol}(W) &= \iiint_W 1 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= \int_{-c}^{c} \left(\iint_{W_z} 1 \,\mathrm{d}x\mathrm{d}y \right) \mathrm{d}z.
\end{aligned}
$$

The area of W_z is $\pi ab(1 - (z/c)^2)$. Hence

$$
\begin{aligned}
\mathrm{vol}(W) &= \int_{-c}^{c} \left(\pi ab(1 - (z/c)^2)\right) \mathrm{d}z \\
&= \pi ab \int_{-c}^{c} \left(1 - (z/c)^2\right) \mathrm{d}z
\end{aligned}
$$

We will change variables $u = z/c$. Then

$$
\begin{aligned}
\mathrm{vol}(W) &= \pi ab \int_{-1}^{1} \left(1 - u^2\right) c \,\mathrm{d}u \\
&= \pi abc \left[u - u^3/3\right]_{-1}^{1} \\
&= \frac{4\pi}{3} abc
\end{aligned}
$$