Line integrals
Based on lecture notes by James McKernan

Let I be an open interval and let
$$\vec{r}\colon I \longrightarrow \mathbb{R}^n,$$
be a parametrised differentiable curve. If $[a,b] \subset I$ then let $C = \vec{r}([a,b])$ be the image of $[a,b]$ and let $f\colon C \longrightarrow \mathbb{R}$ be a function.

Definition 1. *The* ***line integral*** *of f along C is*
$$\oint_C f \,\mathrm{d}s = \int_a^b f(\vec{r}(u))\|\vec{r}'(u)\|\,\mathrm{d}u.$$

Let $u\colon J \longrightarrow I$ be a diffeomorphism between two open intervals. Suppose that u is C^1. We think of u as a coordinate transformation $u = u(t)$; we want to transform from the variable u to the variable t.

Definition 2. *We say that u is* ***orientation-preserving*** *if $u'(t) > 0$ for every $t \in J$.*

We say that u is ***orientation-reversing*** *if $u'(t) < 0$ for every $t \in J$.*

Notice that u is always either orientation-preserving or orientation-reversing (this is a consequence of the intermediate value theorem, applied to the continuous function $u'(t)$).

Define a function
$$\vec{y}\colon J \longrightarrow \mathbb{R}^n,$$
by composition,
$$\vec{y}(t) = \vec{r}(u(t)),$$
so that $\vec{y} = \vec{r} \circ u$.

Now suppose that $u([c,d]) = [a,b]$. Then $C = \vec{y}([c,d])$, so that $\vec{y}$ gives another parametrisation of C.

Lemma 3.
$$\int_a^b f(\vec{r}(u))\|\vec{r}'(u)\|\,\mathrm{d}u = \int_c^d f(\vec{y}(t))\|\vec{y}'(t)\|\,\mathrm{d}t.$$

Proof. We deal with the case that u is orientation-preserving. The case that u is orientation-reversing is similar.

As u is orientation-preserving, we have $u(c) = a$ and $u(d) = b$ and so,
$$\begin{aligned}\int_c^d f(\vec{y}(t))\|\vec{y}'(t)\|\,\mathrm{d}t &= \int_c^d f(\vec{r}(u(t)))\|u'(t)\vec{r}'(u(t))\|\,\mathrm{d}t \\ &= \int_c^d f(\vec{r}(u(t)))\|\vec{r}'(u(t))\|u'(t)\,\mathrm{d}t \\ &= \int_a^b f(\vec{r}(u))\|\vec{r}'(u)\|\,\mathrm{d}u\end{aligned}$$

□

Now suppose that we have a vector field on C,
$$\vec{F}\colon C \longrightarrow \mathbb{R}^n.$$

Definition 4. *The **line integral** of $\vec{F}$ along C is*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_a^b \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u)\, \mathrm{d}u.$$

Note that now the orientation is very important:

Lemma 5.

$$\int_a^b \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u)\, \mathrm{d}u = \begin{cases} \int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t)\, \mathrm{d}t & u'(t) > 0 \\ -\int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t)\, \mathrm{d}t & u'(t) < 0 \end{cases}$$

Proof. We deal with the case that u is orientation-reversing. The case that u is orientation-preserving is similar and easier.

As u is orientation-reversing, we have $u(c) = b$ and $u(d) = a$ and so,

$$\begin{aligned} \int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t)\, \mathrm{d}t &= \int_c^d \vec{F}(\vec{r}(u(t))) \cdot \vec{r}'(u(t))u'(t)\, \mathrm{d}t \\ &= \int_b^a \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u)\, \mathrm{d}u \\ &= -\int_a^b \vec{F}(\vec{r}(u)) \cdot \vec{r}'(u)\, \mathrm{d}u. \end{aligned}$$

□

Example 6. *If C is a piece of wire and $f(\vec{r})$ is the mass density at $\vec{r} \in C$, then the line integral*

$$\int_C f\, \mathrm{d}s,$$

is the total mass of the curve. Clearly this is always positive, whichever way you parametrise the curve.

Example 7. *If C is an oriented path and $\vec{F}(\vec{r})$ is a force field, then the line integral*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{s},$$

is the work done when moving along C. If we reverse the orientation, then the sign flips. For example, imagine C is a spiral staircase and $\vec{F}$ is the force due to gravity. Going up the staircase costs energy and going down we gain energy.