Green's Theorem

Based on lecture notes by James McKernan and Pavel Etingof

Theorem 1 (Green's Theorem)**.** *Let D be a compact subset of $\mathbb{R}^2$ whose boundary ∂D consists of finitely many simple closed curves, where each curve consists of finitely many $\mathcal{C}^1$ arcs, and is given by $\vec{r}_j(t)$, $\vec{r}_j'(t) \neq 0$. Orient the boundary curves so that D is always on the left.*

Let $\vec{F}\colon D \to \mathbb{R}^2$, $\vec{F} = (F_1(x,y), F_2(x,y))$ be a $\mathcal{C}^1$ vector field. Then

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \oint_{\partial D} \vec{F} \cdot \mathrm{d}\vec{s}$$

This was proved in 1828 by George Green, in a privately published paper. He was 35 then, had only one year of schooling, and had worked in a mill in the previous 20 years. It is not known how he had learned contemporary mathematics.

Proof after the next example.

Example 2. *Let $\vec{F}(x,y) = \frac{1}{2}(-y,x)$. Then Green's theorem implies*

$$\text{area}(D) = \oint_{\partial D} \vec{F} \cdot \mathrm{d}\vec{s}.$$

Proof of Green's Theorem for elementary regions of type 1 and 2. Let D be both of type 1 and type 2:

$$\begin{aligned} D &= \{(x,y) \,:\, a \leq x \leq b, u(x) \leq y \leq w(x)\} \\ &= \{(x,y) \,:\, u \leq y \leq w, a(y) \leq x \leq b(y)\}. \end{aligned}$$

We first think of D as being of type 1, and parametrize the boundary in four parts: $\vec{r}_A(t) = (t, c(t))$, for $t \in [a,b]$, $\vec{r}_B(t) = (b,t)$, for $t \in [u(b), w(b)]$, and so on.

Then

$$\begin{aligned} \vec{r}_A'(t) &= (1, u'(x)) \\ \vec{r}_B'(t) &= (0,1) \\ \vec{r}_C'(t) &= (-1, -w'(x)) \\ \vec{r}_D'(t) &= (0,-1). \end{aligned}$$

Note that

$$\oint_{\partial D} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t = \oint_{\partial D} F_1(\vec{r}(t)) r_1'(t)\,\mathrm{d}t + \oint_{\partial D} F_2(\vec{r}(t)) r_2'(t)\,\mathrm{d}t.$$

Now

$$\oint_{\partial D} F_1(r(t)) r_1'(t)\,\mathrm{d}t = \int_a^b F_1(x, u(x))\,\mathrm{d}x - \int_a^b F_1(x, w(x))\,\mathrm{d}x.$$

Using the same technique and thinking of D as being type 2 yields

$$\oint_{\partial D} F_2(r(t)) r_2'(t)\,\mathrm{d}t = \int_u^w F_2(y, a(y))\,\mathrm{d}y - \int_u^w F_2(y, b(y))\,\mathrm{d}y.$$

Hence

$$\oint_{\partial D} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,\mathrm{d}t = \int_a^b \left[F_1(x, u(x)) - F_1(x, w(x))\right]\mathrm{d}x + \int_u^w \left[F_2(y, a(y)) - F_2(y, b(y))\right]\mathrm{d}y$$

Note that

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \iint_D \frac{\partial F_2}{\partial x}\,\mathrm{d}x\mathrm{d}y + \iint_D -\frac{\partial F_1}{\partial y}\,\mathrm{d}x\mathrm{d}y$$

Now, again thinking of D as being type 1, we can do the second integral to get

$$\begin{aligned}\iint_D -\frac{\partial F_1}{\partial y}\,\mathrm{d}x\mathrm{d}y &= -\int_a^b \int_{u(x)}^{w(x)} \frac{\partial F_1}{\partial y}(x,y)\,\mathrm{d}y\mathrm{d}x \\ &= \int_a^b \left[F_1(x,u(x)) - F_1(x,w(x))\right]\,\mathrm{d}x.\end{aligned}$$

Likewise, thinking of D as being of type 2, we can do the first integral to get

$$\iint_D \frac{\partial F_2}{\partial x}\,\mathrm{d}x\mathrm{d}y = \int_u^w \left[F_2(y,a(y)) - F_2(y,b(y))\right]\,\mathrm{d}y,$$

and the claim is proved. □