

THE CROSS PRODUCT
BASED ON LECTURE NOTES BY JAMES MCKERNAN

Blackboard 1. Let $\vec{v}, \vec{w} \in \mathbb{R}^3$, with angle θ between them. The **cross product** of \vec{v} and \vec{w} , denoted $\vec{v} \times \vec{w}$, is the vector defined as follows:

- The length of $\vec{v} \times \vec{w}$ is $\|\vec{v}\|\|\vec{w}\|\sin\theta$.
- $\vec{v} \times \vec{w}$ is orthogonal to both \vec{v} and \vec{w} .
- The three vectors \vec{v} , \vec{w} and $\vec{v} \times \vec{w}$ form a right-handed set of vectors.

Note that $\|\vec{v}\|\|\vec{w}\|\sin\theta$ is the area of the parallelogram with sides \vec{v} and \vec{w} . The cross product only makes sense in \mathbb{R}^3 .

Example 2. We have

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i} \quad \text{and} \quad \hat{k} \times \hat{i} = \hat{j}.$$

By contrast

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i} \quad \text{and} \quad \hat{i} \times \hat{k} = -\hat{j}.$$

Theorem 3. Let \vec{u}, \vec{v} and \vec{w} be three vectors in \mathbb{R}^3 and let λ be a scalar.

- (1) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
- (2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
- (3) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.
- (4) $\lambda(\vec{v} \times \vec{w}) = (\lambda\vec{v}) \times \vec{w} = \vec{v} \times (\lambda\vec{w})$.

Before we prove Theorem 3, let's draw some conclusions from these properties.

Note that (1) of Theorem 3 is what really distinguishes the cross product (the cross product is skew commutative). Note that if \vec{v} and \vec{w} are parallel, then the cross product is the zero vector. One can see this directly from the formula; the area of the parallelogram is zero and the only vector of zero length is the zero vector. On the other hand,

Blackboard 4. If \vec{v} and \vec{w} are parallel, then $\vec{w} = \lambda\vec{v}$. In this case,

$$\begin{aligned} \vec{v} \times \vec{w} &= \vec{v} \times (\lambda\vec{v}) \\ &= \lambda\vec{v} \times \vec{v} \\ &= -\lambda\vec{v} \times \vec{v}. \end{aligned}$$

Hence $\lambda = 0$.

To get from the second to the third line, we just switched the factors. But the only vector which is equal to its inverse is the zero vector.

The cross product is not associative: Consider computing the cross product of \hat{i} , \hat{i} and \hat{j} .

Blackboard 5. On the one hand,

$$(\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \vec{j} = \vec{0}.$$

On the other hand,

$$\hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}.$$

In other words, the order in which we compute the cross product is important. Let's try to compute the cross product using (3).

Blackboard 6. If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, then

$$\begin{aligned}\vec{v} \times \vec{w} &= (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \times (w_1\hat{i} + w_2\hat{j} + w_3\hat{k}) \\ &= v_1w_1(\hat{i} \times \hat{i}) + v_1w_2(\hat{i} \times \hat{j}) + v_1w_3(\hat{i} \times \hat{k}) \\ &\quad + v_2w_1(\hat{j} \times \hat{i}) + v_2w_2(\hat{j} \times \hat{j}) + v_2w_3(\hat{j} \times \hat{k}) \\ &\quad + v_3w_1(\hat{k} \times \hat{i}) + v_3w_2(\hat{k} \times \hat{j}) + v_3w_3(\hat{k} \times \hat{k}) \\ &= (v_2w_3 - v_3w_2)\hat{i} + (v_3w_1 - v_1w_3)\hat{j} + (v_1w_2 - v_2w_1)\hat{k}.\end{aligned}$$

Blackboard 7. A matrix $A = (a_{ij})$ is a rectangular array of numbers, where a_{ij} is in the i row and j th column. If A has m rows and n columns, then we say that A is a $m \times n$ matrix.

Example 8.

$$A = \begin{pmatrix} -2 & 1 & -7 \\ 0 & 2 & -4 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

is a 2×3 matrix. $a_{23} = -4$.

Blackboard 9. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix, then the **determinant** of A is the scalar

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is a 3×3 matrix, then the **determinant** of A is the scalar

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Blackboard 10. The cross product of \vec{v} and \vec{w} is the (formal) determinant

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Let's now turn to the proof of (3).

Blackboard 11. Let \vec{u} , \vec{v} and \vec{w} be three vectors in \mathbb{R}^3 . The **triple scalar product** is $(\vec{u} \times \vec{v}) \cdot \vec{w}$.

The triple scalar product is the *signed* volume of the parallelepiped formed using the three vectors, \vec{u} , \vec{v} and \vec{w} . Indeed, the volume of the parallelepiped is the area of the base times the height. For the base, we take the parallelogram with sides \vec{u} and \vec{v} . The magnitude of $\vec{u} \times \vec{v}$ is the area of this parallelogram. The height of the parallelepiped, up to sign, is the length of \vec{w} times the cosine of the angle, let's call this ϕ , between $\vec{u} \times \vec{v}$ and \vec{w} . The sign is positive, if \vec{u} , \vec{v} and \vec{w} form a *right-handed set* and negative if they form a *left-handed set*.

Lemma 12. If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are three vectors in \mathbb{R}^3 , then

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example 13. The scalar triple product of \hat{i} , \hat{j} and \hat{k} is one. One way to see this is geometrically; the parallelepiped determined by these three vectors is the unit cube, which has volume 1, and these vectors form a right-handed set, so that the sign is positive.

Another way to see this is to compute directly

$$(\hat{i} \times \hat{j}) \cdot \hat{k} = \hat{k} \cdot \hat{k} = 1.$$

Finally one can use determinants,

$$(\hat{i} \times \hat{j}) \cdot \hat{k} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Lemma 14. Let \vec{u} , \vec{v} and \vec{w} be three vectors in \mathbb{R}^3 .

Then

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}.$$

Proof. In fact all three numbers have the same absolute value, namely the volume of the parallelepiped with sides \vec{u} , \vec{v} and \vec{w} . On the other hand, if \vec{u} , \vec{v} , and \vec{w} is a right-handed set, then so is \vec{v} , \vec{w} and \vec{u} and vice-versa, so all three numbers have the same sign as well. \square

Lemma 15. Let \vec{v} and \vec{w} be two vectors in \mathbb{R}^3 .

Then $\vec{v} = \vec{w}$ if and only if $\vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x}$, for every vector \vec{x} in \mathbb{R}^3 .

Proof. One direction is clear; if $\vec{v} = \vec{w}$, then $\vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x}$ for any vector \vec{x} .

So, suppose that we know that $\vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x}$, for every vector \vec{x} . Suppose that $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$. If we take $\vec{x} = \hat{i}$, then we see that

$$v_1 = \vec{v} \cdot \hat{i} = \vec{w} \cdot \hat{i} = w_1.$$

Similarly, if we take $\vec{x} = \hat{j}$ and $\vec{x} = \hat{k}$, then we also get

$$v_2 = \vec{v} \cdot \hat{j} = \vec{w} \cdot \hat{j} = w_2,$$

and

$$v_3 = \vec{v} \cdot \hat{k} = \vec{w} \cdot \hat{k} = w_3.$$

But then $\vec{v} = \vec{w}$ as they have the same components. \square

Proof of Theorem 3. We first prove (1). Both sides have the same magnitude, namely the area of the parallelogram with sides \vec{v} and \vec{w} . Further both sides are orthogonal to \vec{v} and \vec{w} , so the only thing to check is the change in sign.

As \vec{v} , \vec{w} and $\vec{v} \times \vec{w}$ form a right-handed triple, it follows that \vec{w} , \vec{v} and $\vec{v} \times \vec{w}$ form a left-handed triple. But then \vec{w} , \vec{v} and $-\vec{v} \times \vec{w}$ form a right-handed triple. It follows that

$$\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}.$$

This is (1).

To check (2), we check that

$$(\vec{u} \times (\vec{v} + \vec{w})) \cdot \vec{x} = (\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{x},$$

for an arbitrary vector \vec{x} . We attack the LHS. By Lemma 14, we have

$$\begin{aligned} (\vec{u} \times (\vec{v} + \vec{w})) \cdot \vec{x} &= (\vec{x} \times \vec{u}) \cdot (\vec{v} + \vec{w}) \\ &= (\vec{x} \times \vec{u}) \cdot \vec{v} + (\vec{x} \times \vec{u}) \cdot \vec{w} \\ &= (\vec{u} \times \vec{v}) \cdot \vec{x} + (\vec{u} \times \vec{w}) \cdot \vec{x} \\ &= (\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{x}. \end{aligned}$$

This is (2).

We could check (3) by a similar argument. Here is another way.

$$\begin{aligned} (\vec{u} + \vec{v}) \times \vec{w} &= -\vec{w} \times (\vec{u} + \vec{v}) \\ &= -\vec{w} \times \vec{u} - \vec{w} \times \vec{v} \\ &= \vec{u} \times \vec{w} + \vec{v} \times \vec{w}. \end{aligned}$$

This is (3).

To prove (4), it suffices to prove the first equality, since the fact that the first term is equal to the third term follows by a similar derivation. If $\lambda = 0$, then both sides are the zero vector, and there is nothing to prove. So we may assume that $\lambda \neq 0$. Note first that the magnitude of both sides is the area of the parallelogram with sides $\lambda\vec{v}$ and \vec{w} . Hence it remains to be shown that the direction is identical.

If $\lambda > 0$, then \vec{v} and $\lambda\vec{v}$ point in the same direction. Similarly $\vec{v} \times \vec{w}$ and $\lambda(\vec{v} \times \vec{w})$ point in the same direction. As \vec{v} , \vec{w} and $\vec{v} \times \vec{w}$ form a right-handed set, then so do $\lambda\vec{v}$, \vec{w} and $\lambda(\vec{v} \times \vec{w})$. Hence the direction is correct.

If $\lambda < 0$, then \vec{v} and $\lambda\vec{v}$ point in the opposite direction. Similarly $\vec{v} \times \vec{w}$ and $\lambda(\vec{v} \times \vec{w})$ point in the opposite direction. But then $\lambda\vec{v}$, \vec{w} and $\lambda(\vec{v} \times \vec{w})$ still form a right-handed set. So we have shown (4). \square