Surface integrals
Based on lecture notes by James McKernan

Definition 1. *Let $D \subseteq \mathbb{R}^2$, and let $\vec{g}\colon D \to \mathbb{R}^3$ be $\mathcal{C}^1$. Then*

$$\vec{g}(D) = \{(x,y,z) \in \mathbb{R}^3 \,:\, (x,y,z) = \vec{g}(s,t) \text{ for some } (s,t) \in D\}$$

is called a parametrized two dimensional surface.

Example 2. *We can parametrise the sphere,*

$$M = \{\, (x,y,z) \,|\, x^2 + y^2 + z^2 = a^2 \,\},$$

as follows. Let

$$D = [0,\pi) \times [0, 2\pi),$$

and let

$$\vec{g}\colon D \longrightarrow M,$$

be the function

$$\vec{g}(\phi,\theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi).$$

Example 3. *Consider the torus obtained by rotating around the z-axis the circle $y = 0$, $(x-a)^2 + z^2 = b^2$, with $a > b$. In cylindrical coordinates, it is the set of points satisfying $(r-a)^2 + z^2 = b^2$. We can parametrize it by*

$$r = a + b\cos t \qquad z = b\sin t \qquad \theta = s,$$

with $D = [0,2\pi) * [0,2\pi)$. In Cartesian coordinates this becomes

$$x = (a + b\cos t)\cos s \qquad y = b\sin t\cos s \qquad z = b\sin t,$$

or

$$\vec{g}(s,t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t).$$

Let $M = \vec{g}(D)$ be a parametrized two dimensional surface. We can define two tangent vectors, which span the tangent plane to M at $p = \vec{g}(s_0, t_0)$:

$$\vec{T}_s(s_0,t_0) = \frac{\partial\vec{g}}{\partial s}(s_0,t_0)$$
$$\vec{T}_t(s_0,t_0) = \frac{\partial\vec{g}}{\partial t}(s_0,t_0).$$

We get an element of area on M,

$$\mathrm{d}S = \|\vec{T}_s \times \vec{T}_t\|\,\mathrm{d}s\,\mathrm{d}t.$$

Using this we can define the area of M to be

$$\operatorname{area}(M) = \iint_M \mathrm{d}S = \iint_D \|\vec{T}_s \times \vec{T}_t\|\,\mathrm{d}s\,\mathrm{d}t.$$

Example 4. *Let's calculate the tangent vectors for the torus:*

$$\vec{T}_s = \frac{\partial\vec{g}}{\partial s} = (-(a + b\cos t)\sin s, (a + b\cos t)\cos s, 0),$$
$$\vec{T}_t = \frac{\partial\vec{g}}{\partial t} = (-b\sin t\cos s, -b\sin t\sin s, b\cos t).$$

So

$$\vec{T}_s \times \vec{T}_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -(a+b\cos t)\sin s & (a+b\cos t)\cos s & 0 \\ -b\sin t\cos s & -b\sin t\sin s & b\cos t \end{vmatrix}$$
$$= (a+b\cos t)b\cos s\cos t\hat{\imath} + (a+b\cos t)b\sin s\cos t\hat{\jmath} + (a+b\cos t)b\sin t\hat{k}.$$

Therefore,

$$\begin{aligned} \|\vec{T}_s \times \vec{T}_t\| &= (a+b\cos t)b(\cos^2 s\cos^2 t + \sin^2 s\cos^2 t + \sin^2 t)^{1/2} \\ &= (a+b\cos t)b. \end{aligned}$$

As $a \geq b$, *note that* $(a+b\cos t)b > 0$. *Hence*

$$\begin{aligned} \operatorname{area}(M) &= \iint_M \mathrm{d}S \\ &= \iint_D \|\vec{T}_s \times \vec{T}_t\| \,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^{2\pi}\int_0^{2\pi} (a+b\cos t)b\,\mathrm{d}s\,\mathrm{d}t \\ &= 2\pi b\int_0^{2\pi}(a+b\cos t)\,\mathrm{d}t \\ &= 4\pi^2 ab. \end{aligned}$$

Notice that this is the surface area of a cylinder of radius b *and height* $2\pi a$, *as expected.*

Example 5. *Recall that the sphere can be parametrized by*

$$\vec{g}(\phi,\theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi).$$

Let's calculate the tangent vectors,

$$\begin{aligned} \vec{T}_\phi &= \frac{\partial\vec{g}}{\partial\phi} = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi), \\ \vec{T}_\theta &= \frac{\partial\vec{g}}{\partial\theta} = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0). \end{aligned}$$

So

$$\vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= a^2\sin^2\phi\cos\theta\hat{\imath} + a^2\sin^2\phi\sin\theta\hat{\jmath} + a^2\cos\phi\sin\phi\hat{k}.$$

Therefore,

$$\begin{aligned} \|\vec{T}_\phi \times \vec{T}_\theta\| &= a^2\sin\phi(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi)^{1/2} \\ &= a^2\sin\phi. \end{aligned}$$

As $0 < \phi < \pi$, note that $a^2 \sin\phi > 0$. Hence

$$\begin{aligned}
\operatorname{area}(M) &= \operatorname{area}(M) \\
&= \iint_M \mathrm{d}S \\
&= \iint_D \|\vec{T}_\phi \times \vec{T}_\theta\| \,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= 2a^2 \int_0^{2\pi} \mathrm{d}t \\
&= 4\pi a^2.
\end{aligned}$$

Notice that this is the surface area of a sphere of radius a.

Let's now suppose that there are two different ways to parametrise the same piece M of the manifold M:

$$\vec{g}\colon D \longrightarrow M \qquad \text{and} \qquad \vec{h}\colon V \longrightarrow M.$$

Let use (u, v) coordinates for $D \subset \mathbb{R}^2$ and (s, t) coordinates for $V \subset \mathbb{R}^2$. Then

$$\vec{f} = (\vec{h})^{-1} \circ \vec{g}\colon D \longrightarrow V,$$

is a diffeomorphism. Note that $\vec{g} = \vec{h} \circ \vec{f}$. We then have

$$\begin{aligned}
\frac{\partial \vec{g}}{\partial u}(u, v) &= \frac{\partial(\vec{h} \circ \vec{f})}{\partial u}(u, v) \\
&= \frac{\partial \vec{h}}{\partial s}(s, t)\frac{\partial s}{\partial u}(u, v) + \frac{\partial \vec{h}}{\partial t}(s, t)\frac{\partial t}{\partial u}(u, v).
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial \vec{g}}{\partial v}(u, v) &= \frac{\partial(\vec{h} \circ \vec{f})}{\partial v}(u, v) \\
&= \frac{\partial \vec{h}}{\partial s}(s, t)\frac{\partial s}{\partial v}(u, v) + \frac{\partial \vec{h}}{\partial t}(s, t)\frac{\partial t}{\partial v}(u, v).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} &= \left(\frac{\partial \vec{h}}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial \vec{h}}{\partial t}\frac{\partial t}{\partial u}\right) \times \left(\frac{\partial \vec{h}}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial \vec{h}}{\partial t}\frac{\partial t}{\partial v}\right) \\
&= \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t}\left(\frac{\partial s}{\partial u}\frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u}\right) \\
&= \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t}\frac{\partial(s, t)}{\partial(u, v)}.
\end{aligned}$$

It follows that

$$\|\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v}\| = \|\frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t}\|\left|\frac{\partial(s, t)}{\partial(u, v)}\right|.$$

Hence

$$\iint_D \|\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v}\| \,\mathrm{d}u\,\mathrm{d}v = \iint_D \|\frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t}\| |\frac{\partial(s,t)}{\partial(u,v)}| \,\mathrm{d}u\,\mathrm{d}v$$
$$= \iint_V \|\frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t}\| \,\mathrm{d}s\,\mathrm{d}t.$$

Notice that the first term is precisely the integral we use to define the area of M. This formula then says that the area is independent of the choice of parametrisation.