Flux

Based on lecture notes by James McKernan

Let $M \subset \mathbb{R}^3$ be a two dimensional parametrized surface defined by a diffeomorphism $\vec{r}\colon D \to M$, for some $D \subset \mathbb{R}^2$.

Definition 1. *Let $f\colon M \longrightarrow \mathbb{R}$ be a scalar function on M. The integral of f over M is defined as*

$$\iint_M f\,\mathrm{d}S = \iint_D f(\vec{r}(s,t)) \left\| \frac{\partial \vec{r}}{\partial s}(s,t) \times \frac{\partial \vec{r}}{\partial t}(s,t) \right\| \mathrm{d}s\,\mathrm{d}t.$$

We also denote sometimes $\vec{T}_t = \partial\vec{r}/\partial t$ and likewise $\vec{T}_s = \partial\vec{r}/\partial s$.

If we define $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$, then

$$\iint_M f\,\mathrm{d}S = \iint_D f(\vec{r}(s,t)) \sqrt{\left|\frac{\partial(x,y)}{\partial(s,t)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(s,t)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(s,t)}\right|^2}\,\mathrm{d}s\,\mathrm{d}t.$$

Example 2. *Suppose M is the graph of a function $g\colon D \to \mathbb{R}$. Then M is parametrized by $\vec{r}(s,t) = (s,t,g(s,t))$. Hence*

$$\frac{\partial(x,y,z)}{\partial(s,t)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ g_x & g_y \end{pmatrix}$$

and

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} 1 & 0 \\ 0 & 1. \end{pmatrix}$$
$$\frac{\partial(y,z)}{\partial(s,t)} = \begin{pmatrix} 0 & 1 \\ g_x & g_y. \end{pmatrix}$$
$$\frac{\partial(x,z)}{\partial(s,t)} = \begin{pmatrix} 1 & 0 \\ g_x & g_y. \end{pmatrix}.$$

Therefore

$$\iint_M f\,\mathrm{d}S = \iint_D f(s,t,g(s,t)) \sqrt{1 + g_x(s,t)^2 + g_y(s,t)^2}\,\mathrm{d}s\,\mathrm{d}t.$$

Example 3. *The area of a paraboloid $g(x,y) = x^2 + y^2$ over $D = \{(s,t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$ is*

$$\begin{aligned}\iint_D \sqrt{1 + 4s^2 + 4t^2}\,\mathrm{d}s\,\mathrm{d}t &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi \int_0^1 \frac{1}{2}\sqrt{1+4u}\,\mathrm{d}u \\ &= \frac{\pi}{6}\left(5^{3/2} - 1\right).\end{aligned}$$

Definition 4. *Let $\vec{F}\colon M \longrightarrow \mathbb{R}^3$ be a vector field on M. The* flux *of $\vec{F}$ through M is defined by*

$$\iint_M \vec{F} \cdot \mathrm{d}\vec{S} = \iint_D \vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) \mathrm{d}s\,\mathrm{d}t.$$

Let

$$\hat{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\|}.$$

Then

$$\iint_M \vec{F} \cdot \mathrm{d}\vec{S} = \iint_M (\vec{F} \cdot \hat{n}) \, \mathrm{d}S.$$

Note: because $\vec{r}$ is a diffeomorphism, it always points to the "same side" of M.

Example 5. *Find the flux of the vector field given by*

$$\vec{F}(x, y, z) = x\hat{\imath} + 2y\hat{\jmath} + 3z\hat{k},$$

through the triangle M given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \, : \, x \geq 0, y \geq 0, z \geq 0, x + y + z = 1\}.$$

The parametrization is $\vec{r} = (x, y, z) = (s, t, 1 - s - t)$, for

$$D = \{(s, t) \, : \, 0 \leq s \leq 1, 0 \leq t \leq 1 - s\}.$$

Hence

$$\vec{T}_s \times \vec{T}_t = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

and

$$\vec{F}(\vec{r}(s, t)) \cdot (\vec{T}_t(s, t) \times \vec{T}_s(s, t)) = x + 2y + 3z = 3 - 2s - t.$$

Therefore the flux is

$$\begin{aligned} \iint_D (3 - 2s - t) \, \mathrm{d}s \, \mathrm{dt} &= \int_0^1 \int_0^{1-t} (3 - 2s - t) \, \mathrm{d}s \, \mathrm{dt} \\ &= 1. \end{aligned}$$

Example 6. *Recall that the sphere can be parametrized by*

$$\vec{r}(\phi, \theta) = (a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi).$$

Hence the tangent vectors are

$$\begin{aligned} \vec{T}_\phi &= \frac{\partial \vec{r}}{\partial \phi} = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi), \\ \vec{T}_\theta &= \frac{\partial \vec{r}}{\partial \theta} = (-a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0). \end{aligned}$$

and

$$\begin{aligned} \vec{T}_\phi \times \vec{T}_\theta &= a^2 \sin^2 \phi \cos \theta \hat{\imath} + a^2 \sin^2 \phi \sin \theta \hat{\jmath} + a^2 \cos \phi \sin \phi \hat{k} \\ &= a \sin \phi \cdot \vec{r}(\phi, \theta). \end{aligned}$$

Let $\vec{F}(x, y, z)$ be constant: $\vec{F} = (F_1, F_2, F_3)$. Calculate the flux of $\vec{F}$ through the sphere.

The flux is

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\begin{aligned}
\iint_M \vec{F} \cdot \vec{T}_\phi \times \vec{T}_\theta \, \mathrm{d}S &= \int_0^\pi \int_0^{2\pi} \vec{F} \cdot \vec{T}_\phi \times \vec{T}_\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \int_0^\pi a \sin\phi \int_0^{2\pi} \vec{F} \cdot \vec{r}(\phi, \theta) \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= a \int_0^\pi \sin\phi \int_0^{2\pi} (F_1 a \sin\phi \cos\theta + F_2 a \sin\phi \sin\theta + F_3 a \cos\phi) \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= 2\pi a^2 F_3 \int_0^\pi \sin\phi \cos\phi \; \mathrm{d}\phi \\
&= 0.
\end{aligned}
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