STOKES'S THEOREM

BASED ON LECTURE NOTES BY JAMES MCKERNAN AND PAVEL ETINGOF

Theorem 1 (Stokes's Theorem). *Let $S \subset \mathbb{R}^3$ be a differentiable parametrized two dimensional surface. Let $\vec{F}\colon S \longrightarrow \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field.*

Then

$$\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = \int_{\partial S} \vec{F} \cdot \mathrm{d}\vec{s},$$

where ∂S is oriented compatibly with the orientation on S.

Example 2. *Let S look like a pair of pants. Choose the orientation of S such that the normal vector is pointing outwards. There are three oriented curves C_1, C_2 and C_3 (the two legs and the waist). Suppose that we are given a vector field $\vec{B}$ with zero curvature. Then* (1) *says that*

$$\int_{C_3} \vec{B} \cdot \mathrm{d}\vec{s} + \int_{C_1'} \vec{B} \cdot \mathrm{d}\vec{s} + \int_{C_2'} \vec{B} \cdot \mathrm{d}\vec{s} = \iint_S \operatorname{curl} \vec{B} \cdot \mathrm{d}\vec{S} = 0.$$

Here C_1' and C_2' denote the curves C_1 and C_2 with the opposite orientation. In other words,

$$\int_{C_3} \vec{B} \cdot \mathrm{d}\vec{s} = \int_{C_1} \vec{B} \cdot \mathrm{d}\vec{s} + \int_{C_2} \vec{B} \cdot \mathrm{d}\vec{s}.$$

Example 3. *Let S be a closed surface (i.e., without a boundary). For example, the sphere has no boundary. Then Stokes's Theorem implies that*

$$\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = 0.$$

Proof of Stokes's Theorem. Let S be parametrized by $\vec{r}\colon D \to \mathbb{R}^3$ for some region $D \subset \mathbb{R}^2$, and denote $\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$. We define the vector field $\vec{G}\colon D \to \mathbb{R}^2$ by

$$G_1(u, v) = \vec{F}(\vec{r}(u, v)) \cdot \frac{\partial \vec{r}}{\partial u}(u, v) = F_1 x_u + F_2 y_u + F_3 z_u$$

and

$$G_2(u, v) = \vec{F}(\vec{r}(u, v)) \cdot \frac{\partial \vec{r}}{\partial v}(u, v) = F_1 x_v + F_2 y_v + F_3 z_v$$

We claim that

$$\iint_S \operatorname{curl} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_D \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \mathrm{d}u \, \mathrm{d}v, \tag{1}$$

and that

$$\oint_{\partial S} \vec{F} \cdot \mathrm{d}\vec{s} = \oint_{\partial D} \vec{G} \cdot \mathrm{d}\vec{s}. \tag{2}$$

Then the statement follows from Green's Theorem, which states that

$$\oint_{\partial D} \vec{G} \cdot \mathrm{d}\vec{s} = \iint_D \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right) \mathrm{d}u \, \mathrm{d}v.$$

It therefore remains to establish (1) and (2). We will start with the former.

$$\begin{aligned}\operatorname{curl}\vec{F} &= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}\\ &= \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\hat{\imath}-\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)\hat{\jmath}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\hat{k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial\vec{r}}{\partial u} &= \frac{\partial x}{\partial u}\hat{\imath}+\frac{\partial y}{\partial u}\hat{\jmath}+\frac{\partial z}{\partial u}\hat{k}\\ \frac{\partial\vec{r}}{\partial v} &= \frac{\partial x}{\partial v}\hat{\imath}+\frac{\partial y}{\partial v}\hat{\jmath}+\frac{\partial z}{\partial v}\hat{k}.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} &= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}\\ &= \frac{\partial(y,z)}{\partial(u,v)}\hat{\imath}-\frac{\partial(x,z)}{\partial(u,v)}\hat{\jmath}+\frac{\partial(x,y)}{\partial(u,v)}\hat{k}.\end{aligned}$$

So,

$$\operatorname{curl}\vec{F}\cdot\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} = \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)}+\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)\frac{\partial(x,z)}{\partial(u,v)}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)}.$$

On the other hand, if we write out

$$\frac{\partial G_2}{\partial u}-\frac{\partial G_1}{\partial v},$$

using the chain rule, we can see that it is also equal to the RHS. This is (1).

To prove (2), parametrize ∂D by $(u(t),v(t))$, for $t\in[0,1]$. Then

$$\vec{p}(t)=\vec{r}(u(t),v(t))=(x(u(t)),y(u(t)),z(u(t)))$$

is a parametrization of ∂S, and

$$\begin{aligned}&\oint_{\partial S}\vec{F}\cdot\mathrm{d}\vec{S}\\ &= \int_0^1\vec{F}(\vec{p}(t))\cdot\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}(t)\,\mathrm{d}t\\ &= \int_0^1\vec{F}\cdot\left(x_u\frac{\mathrm{d}u}{\mathrm{d}t}+x_v\frac{\mathrm{d}v}{\mathrm{d}t},y_u\frac{\mathrm{d}u}{\mathrm{d}t}+y_v\frac{\mathrm{d}v}{\mathrm{d}t},z_u\frac{\mathrm{d}u}{\mathrm{d}t}+z_v\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}t\\ &= \int_0^1\left((F_1x_u+F_2y_u+F_3z_u)\frac{\mathrm{d}u}{\mathrm{d}t}+(F_1x_v+F_2y_v+F_3z_v)\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}t\\ &= \int_0^1\left(G_1\frac{\mathrm{d}u}{\mathrm{d}t}+G_2\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}t\\ &= \oint_{\partial D}\vec{G}\cdot\mathrm{d}\vec{s}.\end{aligned}$$

This proves (2). □