Calculus of Variations

Example 1. *Consider two circles of radius one, parallel to the z-axis, whose centers are at $(0, 0, a)$ and $(0, 0, -a)$. If we connect them by a cylinder (of radius one), then the surface area of the cylinder is $4\pi a$. Can we connect them by a surface with a smaller area?*

We'll connect them by a surface of cylindrical symmetry whose radius is given by $f\colon [-a, a] \to \mathbb{R}$:

$$S = \{(r, \theta, z) \,:\, 0 \le \theta \le 2\pi, -a \le z \le a, r \le f(z)\}.$$

We will want $f(-a) = f(a) = 1$, so that the ends of the surface coincide with the circle.

The area of this surface is

$$J[f] = \int_{-a}^{a} f(z)\sqrt{1 + f'(z)^2}\,\mathrm{d}z.$$

Example 2. *Consider a ball traveling along a rail $f\colon [0, a] \to \mathbb{R}$, from $x = 0$ to $x = a$, and with $f(0) = f(a) = 0$. If $f(x) < 0$ for $x \in (0, a)$ then the ball will move from $(0, 0)$ to $(a, 0)$, given that there is a constant gravitational field exerting a force of gm downwards.*

The speed of the ball at height $f(x)$ will satisfy $\frac{1}{2}mv(x)^2 = -mgf(x)$, so that $v(x) = \sqrt{-2gf(x)}$. Hence the total travel time will be

$$J[f] = \int_0^a \sqrt{\frac{1 + f'(x)^2}{-2gf(x)}}\,\mathrm{d}x$$

More generally, let $L\colon \mathbb{R}^3 \to \mathbb{R}$ be $\mathcal{C}^2$, let $f\colon [a, b] \to \mathbb{R}$ be $\mathcal{C}^2$, and let

$$J[f] = \int_a^b L(x, f(x), f'(x))\,\mathrm{d}x$$

be a *functional*, or a function from the space of functions to the reals. We would like to minimize (or maximize) J: that is, we would like to find a *function* f such that $J[f]$ is minimal, and which satisfies some condition at a and b (e.g., $f(a) = C_1$ and $f(b) = C_2$ for some constants $C_1, C_2 \in \mathbb{R}$.)

Assume f is a (local) minimum. Let $h\colon [a, b] \to \mathbb{R}$ be a continuous function that satisfies $h(a) = h(b) = 0$. Then for small $\epsilon > 0$, it will hold that $J[f + \epsilon h] \ge J[f]$.

Fix h, and let

$$\Phi(\epsilon) = J[f + \epsilon h].$$

Then Φ has a minimum at $\epsilon = 0$, and $\Phi'(0) = 0$. Hence

$$0 = \Phi'(0) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_a^b L(x, f(x) + \epsilon h(x), f'(x) + \epsilon h'(x))\,\mathrm{d}x.$$

We can move the derivative into the integral to write

$$\begin{aligned} 0 &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} L(x, f(x) + \epsilon h(x), f'(x) + \epsilon h'(x))\,\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial L}{\partial f}(x, f(x), f'(x)) \cdot h + \frac{\partial L}{\partial f'}(x, f(x), f'(x)) \cdot h'(x)\right)\mathrm{d}x, \end{aligned}$$

where $\partial L/\partial f$ and $\partial L/\partial f'$ denote the partial derivatives of L with respect to its second and third argument, respectively. Applying integration by parts to the second addend yields

$$\int_a^b \left(\frac{\partial L}{\partial f} \cdot h - h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}\right) \mathrm{d}x + \frac{\partial L}{\partial f'}h\Big|_a^b = 0,$$

Since h vanishes at a and b, then the last term is zero, and we can write

$$\int_a^b \left(\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'}\right) h \,\mathrm{d}x = 0.$$

Now, this holds for *any choice* of h. We will need the following lemma:

Lemma 3 (Fundamental lemma of the calculus of variations)**.** *Let $g\colon [a,b] \to \mathbb{R}$ in $\mathcal{C}^k$ satisfy*

$$\int_a^b g(x)h(x)\,\mathrm{d}x = 0$$

for all $h\colon [a,b] \to \mathbb{R}$ in $\mathcal{C}^k$ such that $h(a) = h(b) = 0$. Then g is identically zero on $[a,b]$.

Proof. Choose $h(x) = (x-a)\cdot(b-x)\cdot g(x)$. Then

$$\int_a^b (x-a)(x-b)g(x)^2\,\mathrm{d}x = 0.$$

Since the integrand is positive and continuous it must be zero everywhere. Hence g is zero everywhere. □

Applying this above we have that

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'} = 0$$

everywhere on $[a,b]$. This is called the *Euler-Lagrange equation*. Note that is has to hold for all local minima, but may holds for other points too (and not only maxima).

When $L(x, f, f')$ does not depend on x, then this equation can be partially solved to yield the *Beltrami identity*:

$$L - f'\frac{\partial L}{\partial f'} = C,$$

for some constant C.

Let's try to solve the first example. Trying to connect the two circles, we have

$$L(z, f(z), f'(z)) = f(z)\sqrt{1 + f'(z)^2}.$$

The Beltrami identity yields

$$f\sqrt{1+f'^2} - f'\frac{ff'}{\sqrt{1+f'^2}} = C.$$

Hence

$$f\left(1+f'^2\right) - ff'^2 = C\sqrt{1+f'^2}$$

and

$$f = C\sqrt{1+f'^2},$$

or

$$f^2 = C^2 \left(1 + f'^2\right).$$

Solving for f' yields

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\sqrt{f^2 - C^2}}{C}$$

We will solve for z as a function of f:

$$\frac{\mathrm{d}z}{\mathrm{d}f} = \frac{C}{\sqrt{f^2 - C^2}}$$

and so

$$z = C \int \frac{\mathrm{d}f}{\sqrt{f^2 - C^2}} = C \cosh^{-1}(f/C) + D$$

and we have

$$f(z) = C \cosh((z - D)/C).$$

This function is called a *catenary*.

Since $f(-a) = f(a) = 1$ we can find C and D:

$$1 = C \cosh((a - D)/C) = C \cosh((-a - D)/C) = C \cosh((a + D)/C),$$

where the last equality follows from the fact that cosh is an even function. Hence

$$(a - D)/C = (a + D)/C,$$

and $D = 0$. C is therefore the solution to

$$C \cosh(a/C) = 1.$$

This cannot be solved analytically. And it does not always have a solution! In that case there is no continuous function that minimizes the surface area.

Let's try to solve the second example. Here we have

$$L(x, f, f') = \sqrt{\frac{1 + f'^2}{-2gf}}.$$

Applying again the Beltrami identity yields

$$\sqrt{\frac{1 + f'^2}{-2gf}} - \frac{f'^2}{\sqrt{1 + f'^2}} \frac{1}{\sqrt{-2gf}} = C$$

which simplifies to

$$\frac{1}{\sqrt{1 + f'^2}\sqrt{-2gf}} = C.$$

and further to

$$(1 + f'^2)f = -\frac{1}{2gC^2}. \tag{1}$$

Denote $r = \frac{1}{2gC^2}$. Let's parametrize x by θ:

$$x(\theta) = \frac{r}{2}(\theta - \sin\theta),$$

for $\theta \in [0, 2\pi]$, and let $y(\theta) = f(x(\theta))$. Then

$$y'(\theta) = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = f'(x(\theta))\frac{r}{2}(1 - \cos\theta).$$

Hence (1) becomes

$$\left(1 + \frac{4y'(\theta)^2}{r^2(1-\cos\theta)^2}\right) y(\theta) = -r.$$

This can be solved to yield

$$y(\theta) = \frac{r}{2}(\cos\theta - 1).$$

Hence the graph of f is the parametrized curve

$$(x(\theta), y(\theta)) = \frac{r}{2}\big(\theta - \sin\theta, \cos\theta - 1\big).$$

If we set $r/2 = a/(2\pi)$ then this curve passes through $(0,0)$ and $(0,a)$. Hence the solution is

$$(x(\theta), y(\theta)) = \frac{a}{2\pi}\big(\theta - \sin\theta, \cos\theta - 1\big).$$

The lowest point in the curve will be in its middle, at height a/π. This curve is simply a *cycloid*.

Consider a particle moving under the influence of a potential $U\colon \mathbb{R} \to \mathbb{R}$. If we denote its position at time t by $x(t)$, then its kinetic energy is $T(t) = \frac{1}{2}mx'(t)^2$. Let

$$L(t, x, x') = T(t) - U(x) = \tfrac{1}{2}mx'^2 - U(x).$$

The *action* between time t_0 and t_1 is denoted by

$$S = S[x] = \int_{t_0}^{t_1} L(t, x(t), x'(t))\,\mathrm{d}t = \int_{t_0}^{t_1} [T(t) - U(x(t))]\,\mathrm{d}t = \int_{t_0}^{t_1} \left[\tfrac{1}{2}mx'(t)^2 - U(x(t))\right]\,\mathrm{d}t$$

By the Euler-Lagrange equation, every minimal action trajectory satisfies

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x'}$$

and so

$$-\frac{\mathrm{d}U(x)}{\mathrm{d}x} = m\frac{\mathrm{d}x'(t)}{\mathrm{d}t} = mx''(t).$$

Since $F = -\frac{\mathrm{d}U}{\mathrm{d}x}$, this can also be written as

$$F = ma.$$