Planes and distances in $\mathbb{R}^3$
Based on lecture notes by James McKernan

How do we represent a plane Π in $\mathbb{R}^3$? In fact the best way to specify a plane is to give a normal vector $\vec{n}$ to the plane and a point P_0 on the plane. Then if we are given any point P on the plane, the vector $\overrightarrow{P_0P}$ is a vector in the plane, so that it must be orthogonal to the normal vector $\vec{n}$. Algebraically, we have

$$\overrightarrow{P_0P} \cdot \vec{n} = 0.$$

Let's write this out as an explicit equation.

Blackboard 1. *Suppose that the point* $P_0 = (x_0, y_0, z_0)$*,* $P = (x, y, z)$ *and* $\vec{n} = (A, B, C)$*. Then we have*

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0.$$

Expanding, we get

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

We can also write rewrite this as

$$Ax + By + Cz = D.$$

Here

$$D = Ax_0 + By_0 + Cz_0 = (A, B, C) \cdot (x_0, y_0, z_0) = \vec{n} \cdot \overrightarrow{OP_0}.$$

This is perhaps the most common way to write down the equation of a plane.

Example 2. *What is the equation of a plane passing through* $(1, -1, 2)$*, with normal vector* $\vec{n} = (2, 1, -1)$*? We have*

$$(x - 1, y + 1, z - 2) \cdot (2, 1, -1) = 0.$$

So

$$2(x - 1) + y + 1 - (z - 2) = 0,$$

so that in other words,

$$2x + y - z = -1.$$

Example 3.

$$3x - 4y + 2z = 6,$$

is the equation of a plane. A vector normal to the plane is $(3, -4, 2)$.

A line is determined by two points; a plane is determined by three points, provided those points are not collinear (that is, provided they don't lie on the same line). So given three points P_0, P_1 and P_2, what is the equation of the plane Π containing P_0, P_1 and P_2? Well, we would like to find a vector $\vec{n}$ orthogonal to any vector in the plane. Note that $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ are two vectors in the plane, which by assumption are not parallel. The cross product is a vector which is orthogonal to both vectors.

Blackboard 4. *Let* P_0*,*P_1*, and* P_2 *be three vectors in* $\mathbb{R}^3$*. Let*

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}.$$

Then the equation of the plane containing the three points is

$$\overrightarrow{P_0P} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = 0.$$

We can rewrite this a little, using $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$. *Expanding and rearranging gives*

$$\overrightarrow{OP} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = \overrightarrow{OP_0} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}).$$

Note that both sides involve the triple scalar product.

Example 5. *What is the equation of the plane* Π *through the three points,* $P_0 = (1, 1, 1)$, $P_1 = (2, -1, 0)$ *and* $P_2 = (0, -1, -1)$*?*

$$\overrightarrow{P_0P_1} = (1, -2, -1) \qquad \textit{and} \qquad \overrightarrow{P_0P_2} = (-1, -2, -2).$$

Now a vector orthogonal to both of these vectors is given by the cross product:

$$\begin{aligned}
\vec{n} &= \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & -1 \\ -1 & -2 & -2 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} -2 & -1 \\ -2 & -2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -2 \\ -1 & -2 \end{vmatrix} \\
&= 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}.
\end{aligned}$$

Note that

$$\vec{n} \cdot \overrightarrow{P_0P_1} = 2 - 6 + 4 = 0,$$

as expected. It follows that the equation of Π *is*

$$2(x - 1) + 3(y - 1) - 4(z - 1) = 0,$$

so that

$$2x + 3y - 4z = 1.$$

For example, if we plug in $P_2 = (0, -1, -1)$, *then*

$$2 \cdot 0 + 3 \cdot -1 + 4 = 1,$$

as expected.

Example 6. *What is the parametric equation for the line* l *given as the intersection of the two planes* $2x - y + z = 1$ *and* $x + y - z = 2$*?*

Well we need two points on the intersection of these two planes. If we set $z = 0$, *then we get the intersection of two lines in the* xy*-plane,*

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2. \end{aligned}$$

Adding these two equations we get $3x = 3$, *so that* $x = 1$. *It follows that* $y = 1$, *so that* $P_0 = (1, 1, 0)$ *is a point on the line.*

Now suppose that $y = 0$. *Then we get*

$$\begin{aligned} 2x + z &= 1 \\ x - z &= 2. \end{aligned}$$

As before this says $x = 1$ *and so* $z = -1$. *So* $P_1 = (1, 0, -1)$ *is a point on* l.

$$\overrightarrow{P_0P} = t\overrightarrow{P_0P_1},$$

for some parameter t. *Expanding*

$$(x - 1, y - 1, z) = t(0, -1, -1),$$

so that

$$(x, y, z) = (1, 1 - t, -t).$$

We can also calculate distances between planes and points, lines and points, and lines and lines.

Example 7. *What is the distance between the plane $x - 2y + 3z = 4$ and the point $P = (1, 2, 3)$?*

Call the closest point R. Then $\overrightarrow{PR}$ is orthogonal to every vector in the plane, that is, $\overrightarrow{PR}$ is normal to the plane. Note that $\vec{n} = (1, -2, 3)$ is normal to the plane, so that $\overrightarrow{PR}$ is parallel to $\vec{n}$.

Pick any point Q belonging to the plane. Then the triangle PQR has a right angle at R, so that

$$\overrightarrow{PR} = \pm proj_{\vec{n}} \overrightarrow{PQ}.$$

When $x = z = 0$, then $y = -2$, so that $Q = (0, -2, 0)$ is a point on the plane.

$$\overrightarrow{PQ} = (-1, -4, -3).$$

Now

$$\|\vec{n}\|^2 = \vec{n} \cdot \vec{n} = 1^2 + 2^2 + 3^2 = 14 \qquad \textit{and} \qquad \vec{n} \cdot \overrightarrow{PQ} = -2.$$

So

$$proj_{\vec{n}} \overrightarrow{PQ} = \frac{1}{7}(-1, 2, -3).$$

So the distance is

$$\frac{1}{7}\sqrt{14}.$$

Here is another way to proceed. The line through P, pointing in the direction $\vec{n}$, will intersect the plane at the point R. Now this line is given parametrically as

$$(x - 1, y - 2, z - 3) = t(1, -2, 3),$$

so that

$$(x, y, z) = (t + 1, 2 - 2t, 3 + 3t).$$

The point R corresponds to

$$(t + 1) - 2(2 - 2t) + 3(3 + 3t) = 4,$$

so that

$$14t = -2 \qquad \textit{that is} \qquad t = -\frac{1}{7}.$$

So the point R is

$$\frac{1}{7}(6, 16, 18).$$

It follows that

$$\overrightarrow{PR} = \frac{1}{7}(-1, 2, -3),$$

the same answer as before (phew!).

Example 8. *What is the distance between the two lines*

$$(x, y, z) = (t - 2, 3t + 1, 2 - t) \qquad \textit{and} \qquad (x, y, z) = (2t - 1, 2 - 3t, t + 1)?$$

If the two closest points are R and R' then $\overrightarrow{RR'}$ is orthogonal to the direction of both lines. Now the direction of the first line is $(1, 3, -1)$ and the direction of the second line is $(2, -3, 1)$. A vector orthogonal to both is given by the cross product:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -1 \\ 2 & -3 & 1 \end{vmatrix} = -3\hat{\jmath} - 9\hat{k}.$$

To simplify some of the algebra, let's take

$$\vec{n} = \hat{\jmath} + 3\hat{k},$$

which is parallel to the vector above, so that it is still orthogonal to both lines.

It follows that $\overrightarrow{RR'}$ is parallel to $\vec{n}$. Pick any two points P and P' on the two lines. Note that the length of the vector

$$proj_{\vec{n}} \overrightarrow{P'P},$$

is the distance between the two lines.

Now if we plug in $t = 0$ to both lines we get

$$P' = (-2, 1, 2) \qquad \textit{and} \qquad P = (-1, 2, 1).$$

So

$$\overrightarrow{P'P} = (1, 1, -1).$$

Then

$$\|\vec{n}\|^2 = 1^2 + 3^2 = 10 \qquad \textit{and} \qquad \vec{n} \cdot \overrightarrow{P'P} = -2.$$

It follows that

$$proj_{\vec{n}} \overrightarrow{P'P} = \frac{-2}{10}(0, 1, 3) = \frac{-1}{5}(0, 1, 3).$$

and so the distance between the two lines is

$$\frac{1}{5}\sqrt{10}.$$