LIMITS
BASED ON LECTURE NOTES BY JAMES MCKERNAN

Blackboard 1. *Let $q_1, q_2, \dots$ be a sequence of points in $\mathbb{R}^n$. We say that $\lim_n q_n = p$ if for all $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that $|\overrightarrow{q_n p}| < \epsilon$ for all $n \geq m$.*

Blackboard 2. *Let $p \in \mathbb{R}^n$ be a point. The **open ball of radius** $\epsilon > 0$ **around** p is the set*

$$B_\epsilon(p) = \{\, q \in \mathbb{R}^n \mid \|\overrightarrow{pq}\| < \epsilon \,\}.$$

*The **closed ball of radius** $\epsilon > 0$ **about** p is the set*

$$\{\, Q \in \mathbb{R}^n \mid \|\overrightarrow{pq}\| \leq \epsilon \,\}.$$

Blackboard 3. *A subset $A \subset \mathbb{R}^n$ is called **open** if for every $p \in A$ there is an $\epsilon > 0$ such that the open ball of radius ϵ about p is entirely contained in A,*

$$B_\epsilon(p) \subset A.$$

*We say that C is **closed** if the complement of C is open.*

Put differently, an open set is a union of open balls. Open balls are open and closed balls are closed. $[0, 1)$ is neither open nor closed.

Blackboard 4. *Let $A \subset \mathbb{R}^n$. We say that $p \in \mathbb{R}^n$ is an **accumulation point** of A if for every $\epsilon > 0$ the intersection*

$$B_\epsilon(p) \cap (S \setminus \{p\}) \neq \emptyset.$$

In other words, if for every $\epsilon > 0$ there is a point in A that is different than p and is less than ϵ away from p.

This can also be defined as follows:

Blackboard 5. *$p \in \mathbb{R}^n$ is an **accumulation point** of A if there exists a $q_1, q_2, \dots$ with every $q_n \in A \setminus \{p\}$ such that $\lim_n q_n = p$.*

Example 6. *0 is an accumulation point of*

$$\{\, \frac{1}{n} \mid n \in \mathbb{N} \,\} \subset \mathbb{R}.$$

Lemma 7. *A subset $S \subset \mathbb{R}^n$ is closed if and only if S contains all of its accumulation points.*

Example 8. *$\mathbb{R}^n \setminus \{0\}$ is open. One can see this directly from the definition or from the fact that the complement $\{0\}$ is closed.*

Blackboard 9. *Let $A \subset \mathbb{R}^n$ and let $p \in \mathbb{R}^n$ be an accumulation point of A. Suppose that $f \colon A \to \mathbb{R}^m$ is a function.*

We say that f approaches x as q approaches p and write

$$\lim_{q \to p} f(q) = x,$$

*if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $q \in B_\delta(p) \cap A$, $q \neq p$, $f(q) \in B_\epsilon(x)$. In this case we call x the **limit** of f at p.*

It might help to understand the notion of a limit in terms of a game played between two people. Let's call the first player Alice and the second player Bob. Alice wants to prove to Bob that x is the limit of f at q approaches p and Bob is not convinced.

So Bob gets to choose $\epsilon > 0$. It is now up to Alice to choose $\delta > 0$. Then Bob gets to choose a $q \in B_\delta(p) \cap A$, $q \neq p$. Alice wins if $f(q) \in B_\epsilon(x)$, and otherwise Bob wins.

If indeed $\lim_{q\to p} f(q) = x$ then no matter what Bob does, Alice can win. Otherwise, no matter what Alice does, Bob can win.

This can be defined alternatively by

Blackboard 10.

$$\lim_{q\to p} f(q) = x$$

if for every sequence $q_1, q_2, \ldots$, *with* $q_n \in A$ *setminus*$\{p\}$ *such that* $\lim_n q_n = p$ *it holds that* $\lim_n f(q_n) = x$.

Example 11. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) = x^2$. *Show that* $\lim_{q\to 0} f(q) = 0$.

Choose $\epsilon > 0$, *and let* $\delta = \sqrt{\epsilon}$. *Then for every* $q \in (-\delta, \delta) \setminus \{0\}$

$$|f(q) - 0| = q^2 < \delta^2 = \epsilon.$$

Proposition 12. *Let* $f\colon A \to \mathbb{R}^m$ *and* $g\colon A \to \mathbb{R}^m$. *be two functions. Let* $\lambda \in \mathbb{R}$ *be a scalar. If* p *is an accumulation point of* A *and*

$$\lim_{q\to p} f(q) = x \qquad \text{and} \qquad \lim_{q\to p} g(q) = y,$$

then

(1) $\lim_{q\to p}(f + g)(q) = x + y$, *and*
(2) $\lim_{q\to p}(\lambda \cdot f)(q) = \lambda \cdot x$.

Now suppose that $m = 1$.

(3) $\lim_{q\to p}(f \cdot g)(q) = x \ \cdot y$, *and*
(4) *if* $y \neq 0$, *then* $\lim_{Q\to p}(f/g)(q) = x/y$.

Proof. We just prove (1). Suppose that $\epsilon > 0$. As x and y are limits, we may find δ_1 and δ_2 such that, if $\|q - p\| < \delta_1$ and $q \in A \setminus \{p\}$, then $\|f(q) - x\| < \epsilon/2$ and if $\|q - p\| < \delta_2$ and $q \in A \setminus \{p\}$, then $\|g(q) - y\| < \epsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$. If $\|q - p\| < \delta$ and $q \in A \setminus \{p\}$, then

$$\begin{aligned}
\|(f + g)(q) - (x + y)\| &= \|(f(q) - x) + (g(q) - y)\| \\
&\leq \|(f(q) - x)\| + \|(g(q) - y)\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

where we applied the triangle inequality to get from the second line to the third line. This is (1). (2-4) have similar proofs. □

Blackboard 13. *Let* $A \subset \mathbb{R}^n$ *and let* $p \in A$ *be an accumulation point. If* $f\colon A \to \mathbb{R}^m$ *is a function, then we say that* f *is continuous at* p, *if*

$$\lim_{q\to p} f(q) = f(p).$$

We say that f *is continuous, if it is continuous at every point of* A *that is also an accumulation point.*

Theorem 14. *If* $f\colon \mathbb{R}^n \to \mathbb{R}$. *is a polynomial function, then* f *is continuous.*

A similar result holds if $f(x) = P(x)/Q(x)$ is a rational function (a quotient of two polynomials). Its domain is taken to be all the points where Q doesn't vanish.

Example 15. *$f\colon \mathbb{R}^2 \to \mathbb{R}$. given by $f(x,y) = x^2 + y^2$ is continuous.*

Bob likes the following result:

Proposition 16. *Let $A \subset \mathbb{R}^n$ and let $B \subset \mathbb{R}^m$. Let $f\colon A \to B$ and $g\colon B \to \mathbb{R}^l$.*

Suppose that p is an accumulation point of A, x is an accumulation point of B and

$$\lim_{q\to p} f(q) = x \qquad \text{and} \qquad \lim_{y\to x} g(x) = z.$$

Then

$$\lim_{q\to p}[g\circ f](q) = z.$$

Proof. Let $\epsilon > 0$. We may find $\delta > 0$ such that if $\|x - y\| < \delta$, and $y \in B \setminus \{x\}$, then $\|g(y) - z\| < \epsilon$. Given $\delta > 0$ we may find $\eta > 0$ such that if $\|q - p\| < \eta$ and $q \in A \setminus \{p\}$, then $|f(q) - x\| < \eta$. But then if $\|q - p\| < \eta$ and $q \in A \setminus \{p\}$, then $y = f(q) \in B$ and $\|y - x\| < \delta$, so that

$$\begin{aligned}\|[g\circ f](q) - z\| &= \|g(f(q)) - z\| \\ &= \|g(y) - z\| \\ &< \epsilon.\end{aligned}$$

□

Example 17. *Does*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

exist? The answer is no.

To show that the answer is no, we suppose that the limit exists. Suppose we consider restricting to the x-axis. Let

$$f\colon \mathbb{R} \to \mathbb{R}^2,$$

be given by $t \mapsto (t, 0)$. As f is continuous, if we compose we must get a function with a limit,

$$\lim_{t\to 0} \frac{0}{t^2 + 0} = \lim_{t\to 0} 0 = 0.$$

Now suppose that we restrict to the line $y = x$. Now consider the function

$$f\colon \mathbb{R} \to \mathbb{R}^2,$$

be given by $t \mapsto (t, t)$. As f is continuous, if we compose we must get a function with a limit,

$$\lim_{t\to 0} \frac{t^2}{t^2 + t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}.$$

The problem is that the limit along two different lines is different. So the original limit cannot exist.

Example 18. *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2},$$

exist? Let us use polar coordinates. Note that

$$\frac{x^3}{x^2 + y^2} = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta.$$

So we guess the limit is zero.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \left|\frac{x^3}{x^2+y^2}\right| &= \lim_{r\to 0} |r\cos^3\theta| \\ &\le \lim_{r\to 0} |r| = 0. \end{aligned}$$