Derivatives and partial derivatives
Based on lecture notes by James McKernan

The derivative of a function represents the best linear approximation of that function. In one variable, we are looking for the equation of a straight line. We know a point on the line so that we only need to determine the slope.

Blackboard 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a real number. f is **differentiable at** a, with derivative $\lambda \in \mathbb{R}$, if*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lambda.$$

To understand the definition of the derivative of a multi-variable function, it is slightly better to recast (1):

Blackboard 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a real number. f is **differentiable at** a, with derivative $\lambda \in \mathbb{R}$, if*

$$\lim_{x \to a} \frac{f(x) - f(a) - \lambda(x - a)}{x - a} = 0.$$

We are now ready to give the definition of the derivative of a function of more than one variable:

Blackboard 3. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function and let $p \in \mathbb{R}^n$ be a point. f is **differentiable at** p, with derivative the $m \times n$ matrix A, if*

$$\lim_{q \to p} \frac{f(q) - f(p) - A\overrightarrow{pq}}{\|\overrightarrow{pq}\|} = 0.$$

We will write $Df(p) = A$.

So how do we compute the derivative? We want to find the matrix A. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$A\hat{e}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

and

$$A\hat{e}_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

In general, given an $m \times n$ matrix A, we get the jth column of A, simply by multiplying A by the column vector determined by $\hat{e}_j$.

So we want to know what happens if we approach p along the line determined by $\hat{e}_j$. So we take $\overrightarrow{pq} = h\hat{e}_j$, where h goes to zero. In other words, we take $q = p + h\hat{e}_j$. Let's assume that $h > 0$. So we consider the fraction

$$\begin{aligned} \frac{f(q) - f(p) - A(h\hat{e}_j)}{\|\overrightarrow{pq}\|} &= \frac{f(q) - f(p) - A(h\hat{e}_j)}{h} \\ &= \frac{f(q) - f(p) - hA\hat{e}_j}{h} \\ &= \frac{f(q) - f(p)}{h} - A\hat{e}_j. \end{aligned}$$

Taking the limit we get the jth column of A,

$$A\hat{e}_j = \lim_{h \to 0} \frac{f(p + h\hat{e}_j) - f(p)}{h}.$$

Now $f(p + h\hat{e}_j) - f(p)$ is a column vector, whose entry in the ith row is

$$f_i(p+\hat{e}_j) - f_i(p) = f_i(p_1, p_2, \dots, p_{j-1}, p_j + h, p_{j+1}, \dots, p_n) - f_i(p_1, p_2, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_n).$$

and so for the expression on the right, in the ith row, we have

$$\lim_{h \to 0} \frac{f_i(p + h\hat{e}_j) - f_i(p)}{h}.$$

Blackboard 4. *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a function and let $p \in \mathbb{R}^n$. The* ***partial derivative*** *of g at $p = (p_1, \dots, p_n)$, with respect to x_j is the limit*

$$\left.\frac{\partial g}{\partial x_j}\right|_p = \lim_{h \to 0} \frac{g(p_1, p_2, \dots, p_j + h, \dots, p_n) - g(p_1, \dots, p_n)}{h}.$$

Example 5. *Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be the function*

$$g(x, y, z) = x^3 y + x \sin(xz).$$

Then

$$\left.\frac{\partial g}{\partial x}\right|_{(x,y,z)} = 3x^2 y + \sin(xz) + xz\cos(xz),$$

$$\left.\frac{\partial g}{\partial y}\right|_{(x,y,z)} = x^3,$$

and

$$\left.\frac{\partial g}{\partial z}\right|_{(x,y,z)} = x^2 \cos(xz).$$

Putting all of this together, we get

Proposition 6. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function.*

If f is differentiable at p, then $Df(p)$ is the matrix whose (i, j) entry is the partial derivative

$$\left.\frac{\partial f_i}{\partial x_j}\right|_p.$$

Example 7. *Let $f\colon A \to \mathbb{R}^2$ be the function*

$$f(x, y, z) = (x^3 y + x \sin(xz), \log xyz).$$

Here $A \subset \mathbb{R}^3$ is the first octant, the locus where x, y and z are all positive. Supposing that f is differentiable at p, then the derivative is given by the matrix of partial derivatives,

$$Df(p) = \begin{pmatrix} 3x^2 y + \sin(xz) + xz\cos(xz) & x^3 & x^2\cos(xz) \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{pmatrix}.$$

Blackboard 8. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$. be a differentiable function. Then the derivative of f at p, $Df(p)$ is a row vector, which is called the **gradient** of f, and is denoted* $(\nabla f)\Big|_p$,

$$(\frac{\partial f}{\partial x_1}\Big|_p, \dots, \frac{\partial f}{\partial x_n}\Big|_p).$$

The point $(x_1, \dots, x_n, x_{n+1})$ lies on the graph of $f\colon \mathbb{R}^n \to \mathbb{R}$ if and only if $x_{n+1} = f(x_1, \dots, x_n)$.

The point $(x_1, \dots, x_n, x_{n+1})$ lies on the **tangent hyperplane** of $f\colon \mathbb{R}^n \to \mathbb{R}$ at $p = (p_1, \dots, p_n)$ if and only if

$$x_{n+1} = f(p_1, \dots, p_n) + (\nabla f)\Big|_p \cdot (x_1 - p_1, x_2 - p_2, \dots, x_n - p_n).$$

In other words, the vector

$$\left(\frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \dots, \frac{\partial f}{\partial x_n}(p), -1\right),$$

is a normal vector to the tangent hyperplane and of course the point $(p_1, \dots, p_n, f(p_1, \dots, p_n))$ is on the tangent hyperplane.

Example 9. *Let*

$$D = \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \,\},$$

the open ball of radius 1, centred at the origin.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x, y) = \sqrt{1 - x^2 - y^2}.$$

Then

$$\frac{\partial f}{\partial x} = \frac{-2x/2}{\sqrt{1 - x^2 - y^2}} = -\frac{x}{\sqrt{1 - x^2 - y^2}},$$

and so by symmetry,

$$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2}},$$

At the point (a, b), the gradient is

$$(\nabla f)\Big|_{(a,b)} = \frac{-1}{\sqrt{1 - a^2 - b^2}}(a, b).$$

So the equation for the tangent plane is

$$z = f(a, b) - \frac{1}{\sqrt{1 - a^2 - b^2}}(a(x - a) + b(y - b)).$$

For example, if $(a, b) = (0, 0)$, then the tangent plane is

$$z = 1,$$

as expected.