18.022 2014, Homework 3. Due Thursday, October 2nd

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth four points, except the bonus question, which is worth six. Three additional points will be given to any assignment in which there is an honest attempt to answer every question (except perhaps the bonus question).

(1) The *Fibonacci sequence* is the sequence of integers $f_1, f_2, f_3, \ldots$ given by $f_1 = f_2 = 1$ and, for $n > 2$, $f_n = f_{n-1} + f_{n-2}$; each element is the sum of the previous two, so that the sequence starts with $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Let $\vec{v}_n$ be the vector in $\mathbb{R}^2$ given by $\vec{v}_n = (f_n, f_{n+1})$. For example $\vec{v}_1 = (1,1)$, $\vec{v}_2 = (1,2)$ and $\vec{v}_3 = (2,3)$.
   (a) Find a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\vec{v}_n) = \vec{v}_{n+1}$ for all $n \in \mathbb{N}$.
   (b) Note that f is a linear transformation, and write its associated matrix A.
   (c) Explain why, for $n, m > 1$, $\vec{v}_{m+n} = A^n \vec{v}_m$.
   (d) Calculate f_{25}. There no need to show the actual numerical calculation, but write the end result as well as an explanation of how you went about calculating it.

(2) Let $A = \{0,1,2,3,4\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$, and let $E_0 = \{0,2,4,6,\ldots\}$ be the even non-negative numbers.
   (a) What sizes can the finite set B have so that there exists an injective function $f\colon A \to B$? What sizes can the set B have so that there exists a surjective function $f\colon A \to B$? What sizes can the set B have so that there exists a bijection $f\colon A \to B$?
   (b) Let $B = \{a,b,c,d,e\}$. Explain why any injection $f\colon A \to B$ is surjective.
   (c) Let $f\colon \mathbb{N}_0 \to E_0$ be given by $f(n) = 2n$. Is f injective? Surjective?
   (d) Let $g\colon \mathbb{N}_0 \to E_0$ be given by $g(n) = 2n + 2$. Is g injective? Surjective?

(3) Let C be the cone given in spherical coordinates by $\phi = \pi/4$. Let L be the cylinder given in cylindrical coordinates by $r = 1$. Let Π be the plane given in Cartesian coordinates by $z = x + 1$.
   (a) Write Π and $\Pi \cap L$ in cylindrical coordinates.
   (b) Find a bijection $f\colon [0, 2\pi) \to \Pi \cap L$; express it in cylindrical coordinates, and explain why it is indeed a bijection.
   (c) Write Π and $\Pi \cap C$ in spherical coordinates.
   (d) Find a bijection $g\colon (0, 2\pi) \to \Pi \cap C$; express it in spherical coordinates, and explain why it is indeed a bijection.
   (e) Let
$$M = \begin{pmatrix} \cos\pi/4 & 0 & \sin\pi/4 \\ 0 & 1 & 0 \\ -\sin\pi/4 & 0 & \cos\pi/4 \end{pmatrix}.$$
   Explain geometrically the transformation $m\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $m(\vec{v}) = M\vec{v}$. Calculate $m \circ f$ and $m \circ g$, in Cartesian coordinates.

Omer Tamuz. Email: tamuz@mit.edu.

(f) **Bonus question.** Explain why $\Pi \cap L$ is an ellipse and $\Pi \cap C$ is a parabola.

(4) Let P_n be the set of polynomials of degree $n-1$ or less. For example, P_3 is the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a_0 + a_1 x + a_2 x^2$, with $(a_0, a_1, a_2) \in \mathbb{R}^3$. Let $p_n\colon P_n \to \mathbb{R}_n$ be given by

$$p_n(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2).$$

Let $\Delta\colon P_n \to P_{n-1}$ be the function that assigns to each polynomial its derivative. E.g., $\Delta(x^2 + 2x + 2) = 2x + 2$.

(a) Show that p_n is a bijection.

(b) Let $d_n\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be given by

$$d_n = p_{n-1} \circ \Delta \circ p_n^{-1}.$$

Show that d_n is a linear transformation. Is it injective? Surjective? What is the associated matrix of d_5?

(5) Let $\vec{a} \in \mathbb{R}^3$ be non-zero. Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by $f(\vec{v}) = \vec{a} \cdot \vec{v}$. What is the c-level set of f? Describe it geometrically.

(6) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(\vec{0}) = \vec{0}$ and $f(\vec{v}) = \frac{\vec{v}}{|\vec{v}|}$ whenever $\vec{v} \neq \vec{0}$. Show that f is not continuous.

(7) Let $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be given by $f(x,y) = |x|^y$. Let $f'\colon \mathbb{R}^2 \to \mathbb{R}$ agree with f on $\mathbb{R}^2 \setminus \{(0,0)\}$, and take some value at $(0,0)$. Explain why f' is not continuous.

(8) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2|x|$.

(a) Show that f is continuous by directly using the definition of continuity.

(b) Show that $g(x) = |x^3|$ is continuous by using the fact that the composition of continuous functions is continuous.

(9) Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. A function $f\colon A \to B$ is said to be *uniformly continuous* if, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $p \in A$ and every $q \in B_\delta(p)$, $q \neq p$, it holds that $f(q) \in B_\epsilon(f(p))$.

(a) Explain why every uniformly continuous function is continuous.

(b) Describe a game between Al and Betty, where Al has a winning strategy if f is uniformly continuous, and Betty has a winning strategy if f is not uniformly continuous.

(c) Show that $f(x) = 2|x|$ is uniformly continuous.

(d) Let $\bar{\mathbb{N}} = \{1, 1/2, 1/3, \ldots\}$. Show that every function $f\colon \bar{\mathbb{N}} \to \mathbb{R}$ is continuous. Give an example of a function $g\colon \bar{\mathbb{N}} \to \mathbb{R}$ that is not uniformly continuous.