18.022 2014, Homework 7. Due Thursday, October 30^{TH}

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each part of questions 1, 2 and 3 is worth eight points. Question 4 is worth fifteen points. Question 5 is worth two points.

(1) Recall that the following curve describes a helix: $\vec{r}(t) = (a\cos(ct+\theta), a\sin(ct+\theta), bt+z)$ for some constants a, b, c, z, θ.

  (a) Let b, c be constants. Find a vector field $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose flow line through a given point (x, y, z) is $\vec{r}(t) = (r\cos(ct+\theta), r\sin(ct+\theta), bt+z)$, where (r, θ, z) is the representation of (x, y, z) in cylinderical coordinates. Express $\vec{F}(x, y, z)$ in terms of x, y and z (as opposed to cylindrical coordinates).

  (b) Calculate the curl of $\vec{F}$. It $\vec{F}$ rotation free? Explain why $\vec{F}$ is not conservative.

  (c) Are the flow lines of $\vec{F}$ closed? How do these properties of $\vec{F}$ not violate Corollary 7 of lecture 17?

  (d) Calculate the divergence of $\vec{F}$. Is $\vec{F}$ incompressible?

  (e) Explain why there cannot exist an equation expressing the curl of a vector field in terms of the torsion of its flow lines, or vice versa.

(2) A function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is call *holomorphic* if

$$\frac{\partial g_1}{\partial x} = -\frac{\partial g_2}{\partial y}.$$

  (a) Explain why $f\colon \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{C}^2$ is harmonic (i.e, $\nabla^2 f = 0$) iff ∇f is holomorphic.

  (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be in $\mathcal{C}^2$ and by given by $f(x_1, x_2) = h(x_1^2 + x_2^2)$ for some $h\colon [0, \infty] \to \mathbb{R}$. Show that if f is harmonic then it is constant. Hint: show that $h'(z) + zh''(z) = 0$, which is the same as $(zh'(z))' = 0$. Conclude from this that f cannot be defined everywhere on $\mathbb{R}^2$ unless it is constant.

  (c) Now let $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be given by

$$f(x, y) = \log(x^2 + y^2)$$

  Show that f is harmonic (on its domain), even though it is of the form $f(x, y) = h(x^2 + y^2)$.

(3) Suppose $M = (m_{ij})$ is a symmetric 2×2 matrix (i.e., $m_{ij} = m_{ji}$), suppose $\vec{a} \in \mathbb{R}^2$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\vec{x}) = \vec{a} \cdot \vec{x} + \tfrac{1}{2}\vec{x} \cdot (M\vec{x}) + \sin^3(\hat{\imath} \cdot \vec{x}).$$

  This can also be written as

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2 + \tfrac{1}{2}(x_1 m_{11} x_1 + x_1 m_{12} x_2 + x_2 m_{21} x_1 + x_2 m_{22} x_2) + \sin^3(x_1).$$

  Note that (a_i) and (m_{ij}) are constants.

  (a) Calculate the gradient and Hessian of f.

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(b) Calculate $P_{\vec{0},2}f$, the second Taylor polynomial of f, centered at $\vec{0}$. What is the error term?

(c) Calculate $P_{\vec{0},3}f$, the third Taylor polynomial of f.

(4) Read "Lockhart's Lament": `https://www.maa.org/external_archive/devlin/LockhartsLament.pdf`.

What did you disagree/agree with? Do you have any further insights? Please write this part on a separate piece of paper. For those interested: this essay has been extended into a short book which is available for purchase online.

Graders: please give full credit to any answer consisting of at least one coherent and relevant paragraph. Also, please collect these answers separately when returning them to the recitation leaders.

(5) Please offer any suggestions for improvement of this course. These can include the lectures, recitations, problem sets, exams, grading, etc. You may also share any other thoughts or feelings. Please write this part on a separate piece of paper. You may put your name on it, or leave it anonymous.

Graders: please give full credit to any answer consisting of at least one coherent and relevant sentence. Also, please collect these answers separately when returning them to the recitation leaders.