18.022 2014, Homework 9. Due Wednesday, November 12th

Collaboration on homework is encouraged, but individually written solutions are required. Also, it is mandatory to name all collaborators and sources of information on each assignment. Any such named source may be used.

Each question is worth seven points.

(1) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Note that f is continuous. Let $D = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \leq y \leq \pi, y \leq x \leq \pi\}$.

(a) Write D as a type 1 elementary region.

(b) Note that there is no elementary anti-derivative for $\sin x/x$. Still, calculate

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y.$$

(2) The *Expensevier Publishing Company* sells subscriptions to the *Annals of Integration by Parts (AIP)* and the *Journal of Lagrange Multipliers (JLM)*. They have hired the Konman & Schwindler Consulting Company to perform market research. The latter have returned with two functions $f, g\colon [0,C] \to [0,1]$, such that $f(x)$ (respectively, $g(x)$) is the fraction of N potential customers to whom an AIP (respectively, JLM) subscription is worth at least x dollars. They have also discovered that these valuations are "independent": this means that the fraction of potential customers to whom an AIP subscription is worth at least x dollars *and* to whom a JLM subscription is worth at least y dollars is $k(x,y) = f(x)\cdot g(y)$.

Assume that f and g are $\mathcal{C}^1$, so that f' and g' exist and are continuous. Assume that $f(0) = g(0) = 1$ (i.e., a subscription is worth at least zero to everyone) and that $f(C) = g(C) = 0$ (i.e., there is no-one to whom a subscription is worth more than C dollars).

Assume also that an (electronic) subscription costs nothing to produce, so that Expensevier makes a full x dollars for each subscription it sells for x dollars.

As in question 2 of the previous problem set, assume that a customer will buy a product priced at x dollars iff the product it worth at least x to her / him.

You can read about the inspiration for this problem here: `http://en.wikipedia.org/wiki/The_Cost_of_Knowledge`.

(a) Explain why, for $0 \leq a, b \leq C$,

$$k(a,b) = f(a)\cdot g(b) = \iint_S f'(x)\cdot g'(y)\,\mathrm{d}x\mathrm{d}y$$

where $S = [a,C]\times[b,C]$.

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(b) Show that "independent valuations" mean that of the people to whom an AIP subscription is worth at least x dollars, the fraction of those to whom a JLM subscription is worth at least y dollars is still $g(y)$, as it is in the general population.

(c) What is the profit $P(x, y)$ that Expensevier will reap from setting a fixed price of x dollars for AIP and a fixed price of y dollars for JLM?

(d) For $0 \le z \le 2C$, let

$$D_z = \{(x, y) \in [0, C] \times [0, C] \,:\, x + y \ge z\}$$

be the set of price pairs whose sum is at least z. Using the definition of double integrals, give an intuitive explanation (i.e., an informal explanation that is not a proof) for why the total number of customers to whom both subscriptions together are worth at least z dollars is

$$N \iint_{D_z} f'(x) g'(y) \, \mathrm{d}x \mathrm{d}y.$$

(e) Assume henceforth that $C = 1{,}000{,}000$, that $N = 1{,}000$, and that

$$g(x) = f(x) = 1 - x/C.$$

Konman suggests that Expensevier set a fixed price x_m for AIP and a fixed price y_m for JLM. Which prices should Expensevier set to maximize its profits? What will its profits be? (Hint: use question 2 in the previous problem set.)

(f) Schwindler suggests to instead try *bundling*: a customer will only be allowed to buy both subscriptions, for a fixed price z. Assuming $0 \le z \le C$, what bundle price will maximize Expensevier's profit? What will Expensevier's profit be (rounded to the nearest million dollars) at this price, if, as above, C = 1,000,000 and N = 1,000 ?

(3) Let $d \colon [0, 2\pi] \to [0, 1]$ be a continuous function such that $d(0) = d(2\pi)$, let $D_f = \{(r, \theta) \,:\, r \le d(\theta)\}$ be a bounded domain on the plane, given in polar coordinates. Let $h > 0$, and let $W = \{(r, \theta, z) \,:\, 0 \le z \le h, r \le d(\theta) \cdot z/h\}$. Note that W is given in cylindrical coordinates.

The *center of mass* of W is the point (x_c, y_c, z_c), where

$$x_c = \frac{1}{\mathrm{vol}(W)} \iiint_W x \, \mathrm{d}x \mathrm{d}y \mathrm{d}z$$
$$y_c = \frac{1}{\mathrm{vol}(W)} \iiint_W y \, \mathrm{d}x \mathrm{d}y \mathrm{d}z$$
$$z_c = \frac{1}{\mathrm{vol}(W)} \iiint_W z \, \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

(a) If $d(\theta) = 1$, what shape is W? Describe W for general d.

(b) Calculate A, the area of D_f. Express it as a one dimensional integral of an expression involving d.

(c) Calculate $\mathrm{vol}(W)$ in terms of A.

(d) Calculate z_c.

(e) Assume now that $d(\theta + \pi) = d(\theta)$ for all $\theta \in [0, 2\pi]$. Calculate x_c and y_c.

(4) Let $f(x,y) = \sin(x+y)\cos(x-y)$. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk. In this question we will calculate

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

(a) Apply the coordinate transformation $u = \frac{1}{\sqrt{2}}(x+y)$ and $v = \frac{1}{\sqrt{2}}(x-y)$. What is the Jacobian? What is the new domain of the integral, after the coordinate transformation?

(b) Calculate the integral of f on D. Hint: sine is an odd function.