Ma140a, Homework 3
Due Friday, October 23rd

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Consider the sequence of independent random variables $X_1, X_2, \ldots$, where the cumulative distribution function of X_n is given by

$$\mathbb{P}\left[X_n \leq x\right] = \begin{cases} 2^{-n}\exp(2^{-n}x) & \text{if } x < 0 \\ 2^{-n} & \text{if } 0 \leq x < 0.01 \\ 1 & \text{if } 0.01 \leq x. \end{cases}$$

Consider a magical casino in which there is an infinite sequence of slot machines. A gambler starts out with 0 dollars in her account, and proceeds to gamble on each machine in turn. When she gambles on the n^{th} machine her total yield is X_n. Hence her balance at time n is S_n, where $S_0 = 0$ and

$$S_{n+1} = S_n + X_{n+1} = X_1 + \cdots + X_{n+1}.$$

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(a) Prove that if a random variable Y has a cumulative distribution function F, and if $F(x_0) - \lim_{x \nearrow x_0} F(x) = p$, then $\mathbb{P}\left[Y = x_0\right] = p$. Conclude that $\mathbb{P}\left[X_n = 0.01\right] = 1 - 2^{-n}$.

(b) Show that $\mathbb{E}\left[S_n\right] < 0$ for all $n > 0$. Use the definition of expectation and its properties as given in the lecture notes (as opposed to theorems not taught in this course).

(c) Show that $\mathbb{P}\left[\lim S_n = \infty\right] = 1$.

(2) Prove the Dominated Convergence Theorem using the Monotone Convergence Theorem.

(3) Prove that there exists a *simply normal number*: a real number $x \in [0,1]$ such that for any $d > 1$ and $a \in \{0, \ldots, d-1\}$ the digit a occurs in the base d representation of x with asymptotic

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frequency $1/d$:

$$\lim_n \frac{\text{number of times } a \text{ occurs in the first } n \text{ digits of } x}{n} = \frac{1}{d}.$$