MA140A, HOMEWORK 6
DUE FRIDAY, NOVEMBER 20TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) Let $G = (V, E)$ be an undirected, finite graph: $V$ is finite, and $E \subset V \times V$ satisfies $(v, w) \in E$ iff $(w, v) \in E$. Denote by $d(v) = |\{w : (v, w) \in E\}|$ the degree of $v \in V$. Assume that $G$ is connected.

Let $B \subseteq V$ be some non-empty subset of the vertices, which we will refer to as the boundary of the graph. Consider the Markov chain $X_1, X_2, \ldots$ with transition matrix $P$ on the state space $V$ given by

$$P(v, w) = \begin{cases} \frac{1}{d(v)} & \text{if } v \notin B \text{ and } (v, w) \in E \\ 1 & \text{if } v \in B \text{ and } w = v \\ 0 & \text{otherwise.} \end{cases}$$

That is, on $V \setminus B$ the Markov chain moves to adjacent vertices with equal probabilities, and it stops once it reaches $B$.

(a) Let $T_B = \min\{n > 0 : X_n \in B\}$ be the hitting time to $B$. Prove that it is almost surely finite.

(b) Prove that $f(v) = \mathbb{E}_v[T_B]$ is $P$-superharmonic.

(c) Prove that every function $f_B : B \to \mathbb{R}$ has a unique extension to a $P$-harmonic $f : V \to \mathbb{R}$. Hint: use $T_B$.

(d) Suppose $B$ consistes of two vertices: $B = \{b_0, b_1\}$. Let $f_B : B \to \mathbb{R}$ be given by $f_B(b_0) = 0$ and $f_B(b_1) = 1$. Let $f : V \to \mathbb{R}$ be the unique extension of $f_B$ to a $P$-harmonic function. Prove that $f(v) = \mathbb{P}_v[X_{T_B} = b_1]$. That is, that $f(v)$ is the probability that the Markov chain that starts at $v$ hits the boundary at $b_1$, rather than at $b_0$.

(e) Let $K, L$ be two positive integers. A gambler arrives at a casino with $K$ dollars in her pocket. She plays until she runs out of money, or until she has $K + L$ dollars in her

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pocket. At each game she either loses a dollar or gains a dollar, each with probability 1/2. What is the probability that she leaves the casino with $K + L$ dollars?

(2) Let $X$ be a random variable with $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Assume that $x$ is non-atomic, i.e., $\mathbb{P}[X = x] = 0$ for all $x \in \mathbb{R}$. Equivalently, the cumulative distribution function $F_X(x) = \mathbb{P}[X \leq x]$ continous. The median of $X$ is the unique $m \in \mathbb{R}$ such that $F_X(m) = 1/2$. Show that $|\mu - m| \leq \sigma$: the median is at most one standard deviation away from the mean.