Social Sciences Math Camp

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Contents

1 Formal proofs

In this section we will tackle the following questions: What is a formal proof? When we prove something, which arguments do we need to write in full, and which can we let the reader complete? What can we assume is obvious? How do we use lemmas? Learning how to do all this is like learning how to swim: it is impossible to do without active participation.

Exercise 1.1. *Prove that there are infinitely many primes.*

Theorem 1.2 (Pythagoras)**.** $\sqrt{2}$ *is irrational.*

Proof. Assume, by way of contradiction, that $\sqrt{2} = m/n$, where m/n is a reduced fraction. Thus at least one of m or n is odd.

Since $2n^2 = m^2$, m^2 is even, and hence m is even. Thus m^2 is divisible by 4, and so $m^2/2$ is even. Hence $n^2 = m^2/2$ is even, and thus n is also even, and we have reached a contradiction. ☐

Exercise 1.3. *Prove that* $\sqrt{3}$ *is irrational.*

Definition 1.4. Let A,B be sets. A **bijection** from A to B is a function $f: A \to B$ with the property that for every $b \in B$ there is a unique $a \in A$ such that $f(b) = a$.

Definition 1.5. Two sets A,B are said to have the same **cardinality** if there exists a bijection $f: A \to B$.

Definition 1.6. An infinite set A is said to be **countable** if it has the same cardinality as the natural numbers.

Note: this is sometimes refered to as a **countable infinite** set.

Exercise 1.7. *Prove that* $\mathbb{N} \cup \{0\}$ *is countable.*

Exercise 1.8. *Let A be a disjoint union of countably many finite sets. Prove that A is finite or countable.*

Exercise 1.9. *Prove that* $\mathbb{N} \times \mathbb{N}$ *is countable.*

Exercise 1.10. *Let A be the set of sequences in $\mathbb{N} \cup \{0\}$ that are zero from some point on. I.e., the set of sequences $a = (a_1, a_2, \ldots)$ such that there is some N such that $a_n = 0$ for all $n \geq N$. Show that A is countable.*

Theorem 1.11 (Cantor)**.** *If an infinite set A is countable then the set of subsets of A is not countable.*

Proof. Denote by $P(A)$ the set of subsets of A. Assume, by way of contradiction, that there is a bijection $f: \mathbb{N} \to P(A)$. Denote $A_i = f(i)$, so that each A_i is a subset of A, and $P(A) = \{A_1, A_2, \ldots\}$. Since A is countable we can likewise write $A = \{a_1, a_2, \ldots\}$.

Let B be the subset of A given by

$$B = \{a_i \in A : a_i \not\in f(i)\}.$$

Since $B \in P(A)$, and since f is a bijection, there is some $k \in \mathbb{N}$ such that $B = f(k)$. If $a_k \in B$, then, by the definition of B, $a_k \not\in f(k)$. Hence $a_k \not\in B$. But if $a_k \not\in f(k)$ then $a_k \in B$, again by the definition of B, and we have reached a contradiction. □

Exercise 1.12. *Let A be a set. Show there is no bijection between A and the set of subsets of A.*

Exercise 1.13. *A prisoner escapes to the natural numbers. He chooses some $n \in \mathbb{N}$ to hide on the zeroth day. He also chooses some $k \in \mathbb{N}$, and every day hides at a number that is k higher than in the previous day. Hence on day $t \in \{0,1,2,\dots\}$ he hides at $n + k \cdot t$.*

Every day the detective can check one number and see if the prisoner is there. If he is there, she wins. Otherwise she can check again the next day.

Formally, the game played between the prisoner and the detective is the following. The prisoner's strategy space is $\{(n,k) : n,k \in \mathbb{N}\}$, and the detective's strategy space is the set of sequences $(a_0, a_1, a_2, \dots)$ in $\mathbb{N}$. The detective wins if $a_t = n + k \cdot t$ for some t. The prisoner wins otherwise.

Prove that the detective has a winning strategy. That is, prove that there exists a sequence $(a_0, a_1, \dots)$ such that for every (n,k) there is a t with $a_t = n + k \cdot t$.

2 Metrizable Topology

Definition 2.1. A **metric** on a set X is a map $D\colon X\times X\to[0,\infty)$ such that

1. $D(x,y)=0$ iff $x=y$.
2. **Symmetry**. $D(x,y)=D(y,x)$.
3. **Triangle inequality.** $D(x,z)\le D(x,y)+D(y,z)$.

A metric space is the pair (X,D).

Definition 2.2. A **simple, undirected weighted graph** is a triplet (V,E,w). V is the set of vertices. The set of edges E is a subset of $V\times V$ such that $(v_1,v_2)\in E$ implies $(v_2,v_1)\in E$. And $w\colon E\to\mathbb{R}_{++}$ satisfies $w(v_1,v_2)=w(v_2,v_1)$.

A path P from vertex v to vertex u is a sequence of vertices $v=v_1,v_2,\dots,v_k=u$ such that $(v_i,v_{i+1})\in E$. The **length** of a path P is $\ell(P)=\sum_{i=1}^{k-1}w(v_i,v_{i+1})$. Paths of length one is taken to have zero weight.

A graph is **strongly connected** if for every $v,u\in V$ there is a path from v to u. In a strongly connected graph, the distance $D(v,u)$ between vertices v and u is the minimum of $\ell(P)$, where P ranges over all paths from v to u.

Exercise 2.3. *Prove that the distance D is a metric on V.*

Exercise 2.4. *Prove that the following are metrics on $\mathbb{R}^n$.*

1. $D_2(x,y)=\sqrt{\sum_i(x_i-y_i)^2}$.
2. $D_1(x,y)=\sqrt{\sum_i|x_i-y_i|}$.
3.
$$D(x,y)=\begin{cases}1 & \text{if } x\neq y\\ 0 & \text{if } x=y.\end{cases}$$

From here on, when we refer to $\mathbb{R}^n$ (and in particular $\mathbb{R}$) we implicitly refer to the metric space (R^n,D_2).

Denote by $\mathscr{C}=\{0,1\}^{\mathbb{N}}$ the set of all functions from the natural numbers to $\{0,1\}$.

Exercise 2.5. *Prove that $D(x,y)=\inf\{2^{-i}:(x(1),\dots,x(i))=(y(1),\dots,y(i))\}$ is a metric on $\mathscr{C}$.*

We will call $(\mathscr{C},D)$ the **Cantor set**.

Exercise 2.6. *Prove that if $D\colon X\times X\to\mathbb{R}$ satisfies the three properties of a metric then $D(x,y)\ge 0$.*

Exercise 2.7. *Prove that if $D\colon X\times X\to\mathbb{R}$ is a metric then so is $\bar{D}(x,y)=\min\{1,D(x,y)\}$.*

In the following definitions (X,D) is taken to be a metric space.

Definition 2.8. A sequence $(x_n)_n$ in X converges to $x \in X$ (written $\lim_n x_n = x$) if for every $\varepsilon > 0$ there is an $N > 0$ such that $D(x_n,x) < \varepsilon$ for every $n \geq N$.

Definition 2.9. A subset $C \subseteq X$ is **closed** if, when every a sequence $(x_n)_n$ in C converges to x, then $x \in C$. A set is **open** if it is the complement of a closed set.

A metrizable topology on a set X is a collection of open sets obtained from a metric as described above.

Definition 2.10. A sequence $(x_n)_n$ is **Cauchy** if for every ε there exists an N such that $D(x_n,x_m) < \varepsilon$ for all $n,m \geq N$.

Definition 2.11. The open ball of radius ε around $x \in X$ is

$$B_\varepsilon(x) = \{y \in X : D(x,y) < \varepsilon\}.$$

Exercise 2.12. *For x in the Cantor set, what is $B_{2^{-n}}(x)$?*

Exercise 2.13. *Prove that $(x_n)_n$ is Cauchy iff for every ε there is an $x \in X$ such that all but finitely many elements of $\{x_1,x_2,\ldots\}$ are in $B_\varepsilon(x)$.*

Exercise 2.14. *Show that if $(x_n)_n$ converges then it is a Cauchy sequence.*

An example of a sequence that is Cauchy but does not converge is $x_n = 1/n$ in the metric space $X = (0,1)$ with $D(x,y) = |x-y|$.

Exercise 2.15. *Prove that $(x_n)_n$ in the Cantor set is Cauchy iff for every i there is an N such that $x_n(i) = x_N(i)$ for all $n \geq N$.*

This is an important observation, because it shows that the ordering of the coordinates in the definition of the metric does not affect convergence. And neither does the choice of using 2^{-i} in the definition of the metric. Any function of i that is monotone decreasing and goes to zero as i tends to infinity would work.

Exercise 2.16. *Show that if $(x_n)_n$ is a Cauchy sequence that does not converge, then it has no converging subsequences.*

Definition 2.17. A metric space (X,D) is **complete** if every Cauchy sequence in X converges.

So $X = (0,1)$ is not complete, by the example above.

Exercise 2.18. *Prove that the Cantor set is complete.*

Exercise 2.19. *Prove that if a metric space is not complete then it has a sequence with no converging subsequence.*

Exercise 2.20. *Show that every sequence $(x_n)_n$ in the Cantor set has a converging subsequence.*

Recall that a subset of $\mathbb{R}^n$ (with the usual metric D_2) is **bounded** if it is contained in the ball $B_r(0)$ for some $r > 0$.

Definition 2.21. A metric space (X,D) is **totally bounded** if for every ε there is a finite set $B = \{x_1,\dots,x_n\} \subseteq X$ such that the union of $B_\varepsilon(x_1),\dots,B_\varepsilon(x_n)$ is equal to X.

Exercise 2.22. *Prove that a subset of $\mathbb{R}^n$ is bounded if and only if it is totally bounded.*

Exercise 2.23. *Prove that the Cantor set is totally bounded.*

Exercise 2.24. *Prove that if a metric space is not totally bounded then it has a sequence that has no converging subsequence.*

Definition 2.25. A metric space (X,D) is **compact** if every sequence in X has a converging subsequence.

Note: this property is actually called **sequential compactness**, but for metric spaces compactness and sequential compactness are identical.

Exercise 2.26. *Prove that (X,D) is compact iff it is totally bounded and complete.*

Definition 2.27. Let (X,D) and (X',D') be metric spaces. We say that a function $f : X \to X'$ is **continuous** at $x \in X$ if for any sequence $(x_n)_n$ in X that converges to x it holds that $\lim_n f(x_n) = f(x)$. We say that f is continuous if it is continuous at every $x \in X$.

We say that a function $f : X \to \mathbb{R}$ has a maximum if there is some $z \in X$ such that $f(x) \leq f(z)$ for all $x \in X$.

Exercise 2.28. *Prove that if $f : X \to \mathbb{R}$ is continuous, and if X is compact, then f has a maximum.*

Definition 2.29. Let (X,D) be a metric space, and let A be a countable set. The **topology of pointwise convergence** on $\Omega = X^A$ is the topology in which a sequence $(\omega_n)_n$ in Ω converges to ω iff $\omega_n(a)$ converges to $\omega(a)$ for each $a \in A$.

Exercise 2.30. *Find a metric on X^A that induces the topology of pointwise convergence.*

Exercise 2.31. *Under the metric from Exercise 2.30, show that if (X,D) is compact then X^A is also compact.*

Definition 2.32. Let (X,D) be a metric space, and let A be a countable set. The **topology of uniform convergence** on $\Omega = X^A$ is the topology in which a sequence $(\omega_n)_n$ in Ω converges to ω iff for every ε there is an n such that $D(\omega_n(a),\omega(a)) < \varepsilon$ for all $a \in A$.

Exercise 2.33. *Find a metric on X^A that induces the topology of uniform convergence.*

Exercise 2.34. *Under the metric from Exercise 2.33, give an example of a compact (X,D) such that X^A is not compact.*

3 Convexity

Definition 3.1. A subset $A \subseteq \mathbb{R}^n$ is **convex** if for every $x, y \in A$ and $\alpha \in (0,1)$ it holds that $\alpha x + (1-\alpha)y$ is in A.

Exercise 3.2. *What are the convex subsets of* $\mathbb{R}$*?*

Exercise 3.3. *Show that if A is a convex and compact subset of $\mathbb{R}$, and if $f : A \to A$ is continuous, then there is some $a \in A$ such that $f(a) = a$.*

We will not prove this here, but the same statement is true in $\mathbb{R}^n$. This is called the Brouwer fixed point theorem.

Definition 3.4. A **normed vector space** is a vector space X with a norm $\|\cdot\|$ such that

1. $\|x\| \in [0,\infty)$.
2. $\|x\| = 0$ iff $x = 0$.
3. $\|\alpha x\| = \alpha\|x\|$.
4. $\|x + y\| \leq \|x\| + \|y\|$.

Exercise 3.5. *Denote by ℓ^∞ the set of all functions $f : \mathbb{N} \to \mathbb{R}$ for which there exists an M such that $|f(i)| \leq M$ for all $i \in \mathbb{N}$. Let $\|f\| = \max_i |f(i)|$. Show that ℓ^∞ is a normed vector space.*

Exercise 3.6. *Prove that $D(x, y) = \|x - y\|$ is a metric.*

Exercise 3.7. *Prove that multiplication by a scalar and addition are continuous under the topology induced by $D(x, y) = \|x - y\|$. The former is the map from $\mathbb{R} \times X$ to X that maps (α, x) to αx, and the latter is the map from $X \times X$ to X that maps (x, y) to $x + y$.*

Definition 3.8. A subset A of a normed vector space is **convex** if for every $x, y \in A$ and $\alpha \in (0,1)$ it holds that $\alpha x + (1-\alpha)y$ is in A.

Definition 3.9. Let A be a convex subset of a normed vector space. A function $f : A \to \mathbb{R}$ is called **convex** if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in A$ and $\alpha \in (0,1)$. It is called **strictly convex** if the inequality holds strictly whenever $x \neq y$. f is (strictly) **concave** if $-f$ is (strictly) convex.

Exercise 3.10. *Let A be a compact, convex subset of a normed vector space, and let $f : A \to \mathbb{R}$ be strictly concave and continuous. Show that there is a unique $z \in A$ such that $f(x) \leq f(z)$ for all $x \in A$.*

Definition 3.11. A **linear functional** on a normed vector space X is a map $f : X \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$. It is **bounded** if there is some M such that $|f(x)| \leq M\|x\|$ for all x.

Exercise 3.12. *Show that if f is a linear functional on $\mathbb{R}^n$ then f is of the form $f(x) = \sum_{i=1}^n x_i y_i$ for some $y \in \mathbb{R}^n$.*

Exercise 3.13. *Show that a linear functional on normed vector space is bounded iff it is continuous.*

Definition 3.14. Let X be a normed vector space. A **hyperplane** is a subset $H \subseteq X$ that takes the form

$$H = \{x \in X : f(x) = c\}$$

for some $c \in \mathbb{R}$ and bounded linear functional $f : X \to \mathbb{R}$ that is not identically 0.

Exercise 3.15. *Show that a hyperplane in $\mathbb{R}^n$ is of the form*

$$H = \{w + x : w \in W\}$$

for some $x \in \mathbb{R}^n$ and W a vector subspace of $\mathbb{R}^n$.

Exercise 3.16. *Let A be a closed convex subset of $\mathbb{R}^n$, and let $b \in \mathbb{R}^n$ be any point that is not in A. Show that there is an $a \in A$ that minimizes the distance to b, among all points in A.*

Exercise 3.17. *Let A,B be disjoint closed convex subsets of $\mathbb{R}^n$. Then there is a linear functional f on $\mathbb{R}^n$ and a $c \in \mathbb{R}$ such that $f(x) > c$ and $f(y) < c$ for all $x \in A$ and $y \in B$.*

Definition 3.18. A **closed half-space** of a normed vector space X is the set of points $\{x : f(x) \geq c\}$ for some linear functional $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$.

Exercise 3.19. *Every closed convex subset of $\mathbb{R}^n$ is equal to the intersection of all the closed half-spaces that contain it.*

Definition 3.20. Let A be a convex subset of a normed vector space X. A point $x \in X$ is **extreme** if there are no $y \neq z \in X$ and $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha)z$.

Theorem 3.21 (Krein-Milman)**.** *Let A be a compact convex subset of a normed vector space X. Let C be the set of convex combinations of the extreme points of A: i.e., each element of C is of the form $\sum_{i=1}^n \alpha_i x_i$, where $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$ and x_i is extreme. Then A is equal to the closure of C.*

4 Solutions

Exercise 2.5. Given $x, y \in \mathscr{C}$, let $R(x,y) = \{2^{-i} : (x(1),\dots,x(i)) = (y(1),\dots,y(i))\}$, so that $D(x,y) = \inf R(x,y)$.

We verify the three properties of a metric.

1. Since $R(x,x) = \{2^{-1}, 2^{-2}, 2^{-3}, \dots\}$, we have that $D(x,x) = 0$. Suppose $D(x,y) = 0$. Then $R(x,y) = \{2^{-1}, 2^{-2}, 2^{-3}, \dots\}$, and so $x(i) = y(i)$ for all i, and $x = y$.

2. Since $R(x,y) = R(y,x)$, $D(x,y) = D(y,x)$.

3. If 2^{-i} is in both $R(x,y)$ and $R(y,z)$, then

$$(x(1),\dots,x(i) = (y(1),\dots,y(n)) = (z(1),\dots,z(n)),$$

and so $2^{-i} \in R(x,z)$. Hence $R(x,y) \cap R(y,z) \subseteq R(x,z)$, and thus

$$D(x,z) = \inf R(x,z) \le \inf(R(x,y) \cap R(y,z)).$$

Since

$$\inf(R(x,y) \cap R(y,z)) = \max\{\inf R(x,y), \inf R(y,z)\} = \max\{D(x,y), D(y,z)\}$$

we have shown that

$$D(x,z) \le \max\{D(x,y), D(y,z)\},$$

which is, in turn, at most $D(x,y) + D(y,z)$.

Exercise 2.6. By the triangle inequality, $0 = D(x,x) \le D(x,y) + D(y,x)$. By symmetry $D(y,x) = D(x,y)$, and so $0 \le 2D(x,y)$.

Exercise 2.7. We verify the three properties of a metric:

1. $\bar{D}(x,x) = \min\{1, D(x,x)\} = \min\{1,0\} = 0$. Conversely, assume that $\bar{D}(x,y) = 0$. Then $\min\{1, D(x,y)\} = 0$, and so $D(x,y) = 0$ and $x = y$.

2. By the symmetry of D, $\bar{D}(x,y) = \min\{1, D(x,y)\} = \min\{1, D(y,x)\} = \bar{D}(y,x)$.

3. Note that for any $a, b \in \mathbb{R}_+$, $\min\{1, a+b\} \le \min\{1,a\} + \min\{1,b\}$. This holds because if $a+b \le 1$ then both sides of the inequality are equal to $a+b$, and if $a+b > 1$ then the left hand side is 1 while the right hand side is at least 1. It thus follows from the triangle inequality of D that

$$\begin{aligned}\bar{D}(x,z) &= \min\{1, D(x,z)\} \\ &\le \min\{1, D(x,y) + D(y,z)\} \\ &\le \min\{1, D(x,y)\} + \min\{1, D(y,z)\} \\ &= \bar{D}(x,y) + \bar{D}(y,z).\end{aligned}$$

Exercise 2.13. Suppose $(x_n)_n$ is Cauchy. Then for every ε there is an N such that $D(x_n,x_m) < \varepsilon$ for all $n,m \geq N$. In particular $D(x_n,x_N) < \varepsilon$ for all $n \geq N$, and thus all the elements of the sequence except perhaps $x_1,\ldots,x_{N-1}$ are in $B_\varepsilon(x_N)$.

For the other direction, the condition implies that for every $\varepsilon/2$ there is an x such theta $D(x_n,x) < \varepsilon/2$ for all n but finitely many. Hence there exists an N such that $D(x_n,x) < \varepsilon/2$ for all $n \geq N$. Hence, for any $n,m \geq N$ the triangle inequality implies that

$$D(x_n,x_m) \leq D(x_n,x) + D(x,x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and so $(x_n)_n$ is Cauchy.

Exercise 2.14. Let $\lim_n x_n = x$, so that $\lim_n D(x_n,x) = 0$. Then for every $\varepsilon/2$ there is an N such that $D(x_n,x) < \varepsilon/2$ for all $n \geq N$. Therefore, by the triangle inequality, it holds for all $n,m \geq N$ that

$$D(x_n,x_m) \leq D(x_n,x) + D(x,n_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and so $(x_n)_n$ is a Cauchy sequence.

Exercise 2.15. Suppose $(x_n)_n$ is Cauchy. Then for every i there is an N such that $D(x_n,x_N) \leq 2^{-i}$ for all $n \geq N$. By the definition of D, this means that $(x_n(1),\ldots,x_n(i)) = (x_N(1),\ldots,x_N(i))$, and in particular $x_n(i) = x_N(i)$.

Conversely, given ε, choose i so that $2^{-i} < \varepsilon/2$. By assumption, for each $j \leq i$ there is an N_j such that $x_n(i) = x_{N_j}(i)$ for all $n \geq N_j$. Hence for $N = \max\{N_1,\ldots,N_i\}$ it holds that for all $n \geq N$ that $(x_n(1),\ldots,x_n(i)) = (x_N(1),\ldots,x_N(i)$, and thus $D(x_n,x_N) \leq 2^{-i} < \varepsilon/2$. It follows that for $n,m \geq N$ it holds that

$$D(x_n,x_m) \leq D(x_n,x_N) + D(x_m,x_N) < \varepsilon.$$

Exercise 2.16. Let $(x_{n_m})_m$ be a subsequence, and assume towards a contradiction that it converges to some $x \in X$, so that $\lim_m D(x_{n_m},x) = 0$. Then for every $\varepsilon/2 > 0$ there is some N so that $D(x_{n_m},x) < \varepsilon/2$ for all $m \geq N$. Since the sequence is Cauchy, there is some N' so that $D(x_k,x_{n_m}) < \varepsilon/2$ for all $k,n_m \geq N'$. Hence, for all $n \geq \max\{N,N'\}$ it follows from the triangle inequality that

$$D(x_n,x) \leq D(x_n,x_{n_m}) + D(x_{n_m},x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $\lim_n D_n(x_n,x) = 0$, and thus $(x_n)_n$ converges, in contradiction to our assumption. It follows that $(x_n)_n$ does not have any converging subsequences.

Exercise 2.19. Suppose (X,D) is not complete. Then, by definition, it has a Cauchy sequence $(x_n)_n$ that does not converge to any x. By Exercise 2.16 this sequence has no converging subsequences.

Exercise 2.20. Let $(x_n)_n$ be a sequence in the Cantor set. Given $x \in \mathscr{C}$, we will denote by $A(m,x)$ the set of indices n such that $(x_n(1),\dots,x_n(m)) = (x(1),\dots,x(m))$. We set $A(0,x) = \mathbb{N}$.

We first claim that there exists an $x \in \mathscr{C}$ such that $A(m,x)$ is infinite for all m. We construct such an x by induction on m. For $m = 0$, $A(m,x)$ is infinite by definition, for any x. Assume that we have chosen $x(1),\dots,x(m)$ so that $A(0,x),\dots,A(m,x)$ are all infinite. For each $n \in A(m,x)$, either $x_n(m+1) = 1$ or $x_n(m+1) = 0$. Since $A(m,x)$ is infinite, either $x_n(m+1) = 0$ for infinitely many $n \in A(m,x)$, or $x_n(m+1) = 1$ for infinitely many $n \in A(m,x)$ (or both). In the former case choose $x(m+1) = 0$, and otherwise choose $x(m+1) = 1$. We thus have that $A(m+1,x)$ is infinite. Since we did not change $x(1),\dots,x(m)$, we still have that $A(0,x),\dots,A(m,x)$ are infinite. Hence, by induction, we have constructed an x such that $A(m,x)$ is infinite for all m.

Define $n_m = \min A(m,x)$. Since each $A(m,x)$ is infinite it is in particular non-empty, and thus has a minimum. We claim that $\lim_m x_{n_m} = x$. This holds because $n_m \in A(m,x)$, and so x_{n_m} agrees with x on the coordinates 1 through m. Thus $D(x_n,x) \le 2^{-m}$, and $\lim_m D(x_{n_m},x) = 0$.

Exercise 4.1. *Prove that if a metric space is not totally bounded then it has a sequence that has no converging subsequence.*

Exercise 2.24. If (X,D) is not totally bounded then there is some ε such that no finite collection of balls of radius ε covers X. We construct a sequence $(x_n)_n$ that has no converging subsequences.

Let x_1 be any element of x. Given $x_1,\dots,x_n$, let x_{n+1} be any element of x that is not in the union of the balls $B_\varepsilon(x_1),\dots,B_\varepsilon(x_n)$. There is such an element, since X is not the union of any finite set of balls of radius ε. Thus, $D(x_n,x_m) \ge \varepsilon$ for all n,m, and $(x_n)_n$ is not Cauchy. Moreover, no subsequence $(x_{n_m})_m$ can be Cauchy, since the distance between any pair of its elements is again at least ε. Hence, by Exercise 2.14, no subsequence is converging.

Exercise 2.26. If X is not complete or not totally bounded then it is not compact, by exercises 2.19 and 2.24. It thus remains to be shown that if X is totally bounded and complete then it is compact. To this end it suffices to show that every sequence has a Cauchy subsequence, from which convergence will follow by completeness.

For each m, let Y_m be a finite set of balls of radius 2^{-m} whose union is equal to X. Let $(x_n)_n$ be a sequence in X. Similarly to the proof of Exercise 2.20, there is some $B^1 \in Y_1$ that contains infinitely many of the elements of $(x_n)_n$. Then, there some $B^2 \in Y_2$ that intersects B^1, and contains infinitely many of the elements of $(x_n)_n$. Continuing, we arrive at a sequence of balls $(B^m)_m$ with the property that each contains infinitely many elements of $(x_n)_n$, such that the radius of B^m is 2^{-m}, and such that B^m intersects B^{m+1}. Consider the subsequence $(x_{n_m})_m$ where n_m is the minimal index such that $x_{n_m} \in B^m$. By the intersection property of the balls, the distance between any point $x \in B^m$ and $y \in B^k$ is at most $4 \cdot 2^{-\min\{m,k\}}$, as a consequence of repeated applications of the triangle inequality. We thus have that if $m,k \ge N$ then $D(x_{n_m},x_{n_k}) \le 4 \cdot 2^{-N}$, and thus $(x_{n_m})_m$ is a Cauchy sequence.

Exercise 2.28. Denote $f(X) = \{c \in \mathbb{R} : f(x) = c \text{ for some } x \in X\}$.

Let $c = \sup f(X)$, with $c = \infty$ in case $f(X)$ is not bounded from above. Let $(c_n)_n$ be a sequence in $f(X)$ with $\lim_n c_n = c$. Then there exists a sequence $(x_n)_n$ in $\mathbb{R}$ with $f(x_n) = c_n$, and hence $\lim_n f(x_n) = c$. Let $(x_{n_m})_m$ be a converging subsequence, and denote its limit by z. Then, since f is continuous, $\lim_m f(x_{n_m}) = f(z)$, and so $f(z) = c$. Thus $f(x) \le f(z)$ for all $x \in X$, and f has a maximum.

Exercise 2.30. Denote $\bar{D}(x,y) = \min\{1, D(x,y)\}$. Enumerate $A = \{a_1, a_2, \ldots\}$. Let D_Ω be the metric on $\Omega = X^A$ given by

$$D_\Omega(\omega,\theta) = \sum_i 2^{-i}\bar{D}(\omega(a_i),\theta(a_i)).$$

We now show that D_Ω induces the topology of pointwise convergence. That is, that a sequence $(\omega_n)_n$ converges pointwise to ω iff $\lim_n D_\Omega(\omega_n,\omega) = 0$.

Suppose that $(\omega_n)_n$ converges pointwise to ω, i.e., $\lim_n D(\omega_n(a),\omega(a)) = 0$ for all $a \in A$. Then also $\lim_n \bar{D}(\omega_n(a),\omega(a)) = 0$. It follows that for every m there is an N such that $\bar{D}(\omega_n(a),\omega(a)) \le 1/m^2$ for all $a \in \{a_1,\ldots,a_m\}$ and all $n \ge N$. Thus, for all such n,

$$\begin{aligned}
D_\Omega(\omega_n,\omega) &= \sum_i 2^{-i}\bar{D}(\omega_n(a_i),\omega(a_i) \\
&= \sum_{i=1}^m 2^{-i}\bar{D}(\omega_n(a_i),\omega(a_i) + \sum_{i=m+1}^\infty 2^{-i}\bar{D}(\omega_n(a_i),\omega(a_i) \\
&\le \sum_{i=1}^m \frac{1}{m^2} + \sum_{i=m+1}^\infty 2^{-i} \\
&= \frac{1}{m} + 2^{-m},
\end{aligned}$$

and so $\lim_n D_\Omega(\omega_n,\omega) = 0$.

Conversely, suppose that $\lim_n D_\Omega(\omega_n,\omega) = 0$. Then $\lim_n \bar{D}(\omega_n(a_i),\omega(a_i)) = 0$, since $\bar{D}(\omega_n(a_i),\omega(a_i)) \le 2^i D_\Omega(\omega_n,\omega)$. But then also $\lim_n D(\omega_n(a_i),\omega(a_i)) = 0$.

Exercise 2.33. Denote $\bar{D}(x,y) = \min\{1, D(x,y)\}$. Let D_Ω be the metric on $\Omega = X^A$ given by

$$D_\Omega(\omega,\theta) = \min\{\bar{D}(\omega(a_i),\theta(a_i)) : a \in A\}$$

We now show that D_Ω induces the topology of uniform convergence. That is, that a sequence $(\omega_n)_n$ converges uniformly to ω iff $\lim_n D_\Omega(\omega_n,\omega) = 0$.

Suppose that $(\omega_n)_n$ converges uniformly to ω, i.e., for every ε there is an N such that $D(\omega_n(a),\omega(a)) < \varepsilon$ for all $a \in A$ and $n \ge N$. Then $\bar{D}(\omega_n(a),\omega(a)) < \varepsilon$ for all $a \in A$ and $n \ge N$, and hence $D_\Omega(\omega_n,\omega) < \varepsilon$. Thus indeed $\lim_n D_\Omega(\omega_n,\omega) = 0$.

Conversely, suppose that $\lim_n D_\Omega(\omega_n,\omega) = 0$. Then, for every ε there is an N such that $\bar{D}(\omega_n(a),\omega(a)) < \varepsilon$ for all $a \in A$. It follows that, for $\varepsilon < 1$, $D(\omega_n(a),\omega(a)) < \varepsilon$ for all $a \in A$ and $n \ge N$, and so $(\omega_n)_n$ converges uniformly to ω.

Exercise 3.2. Clearly, the empty set is convex, as are the closed intervals $[a,b]$, the open intervals (a,b) and the half open intervals $[a,b)$ and $(a,b]$, where $a,b \in \mathbb{R} \cup \{\infty\}$.

To see that these are all, suppose A is non-empty and convex. Let $a = \inf A$ and $b = \sup A$. Then for every x in (a,b) there are $a', b' \in A$ such that $a' \le x \le b'$, and so $x \in A$. Thus A is one of (a,b), $[a,b]$, $[a,b)$ and $(a,b]$, depending on whether a is in A and / or b is in A.

Exercise 3.10. By compactness and continuity, f has at least one such z. To show uniqueness, suppose $f(z) = f(z')$ with $z' \neq z$. Then, since A is convex, $z/2 + z'/2 \in A$, and by strict concavity,

$$f(z/2 + z'/2) > f(z)/2 + f(z')/2 = f(z),$$

in contradiction to the assumption that $f(x) \le f(z)$ for all $x \in A$.

Exercise 3.12. Let f be a linear functional on $\mathbb{R}^n$. Let $y_i = f(e_i)$, where e_i is the i^{th} vector in the standard basis of $\mathbb{R}^n$. Then $x = \sum_i x_i e_i$, and so, by the linearity of f,

$$f(x) = f\left(\sum_i x_i e_i\right) = \sum_i x_i f(e_i) = \sum_i x_i y_i.$$

Exercise 3.13. Suppose that f is not bounded. Then there is a sequence $(x_n)_n$, with $\|x\| \le 1$, such that $\lim_n f(x_n) = \infty$. By restricting to a subsequence, we may assume that $f(x_n) \ge n$. Let $y_n = x_n/n$. Then, by linearity, $f(y_n) \ge 1$. But $\|y_n\| \le 1/n$, and so $\lim_n y_n = 0$. Since $f(0) = 0$ (by linearity), we have that $\lim_n f(y_n) = 1 \neq f(\lim_n y_n)$, and so f is not continuous.

Suppose that f is bounded. Then there is some M such that $|f(x)| \le M\|x\|$. Let $(x_n)_n$ be a sequence in X that converges to 0, so that $\lim_n \|x_n\| = 0$. Then

$$\lim_n |f(x_n)| \le \lim_n M\|x_n\| = 0,$$

and thus also $\lim_n f(x_n) = 0$. Thus f is cotinuous at 0. Finally, suppose $\lim_n x_n = x$. Then

$$\begin{aligned}
\lim_n f(x_n) &= \lim_n f(x_n) - f(x) + f(x) \\
&= \lim_n f(x_n - x) + f(x) \\
&= f(x) + \lim_n f(x_n - x).
\end{aligned}$$

Since f is continuous at 0, and since $\lim_n x_n - x = 0$, we have that $\lim_n f(x_n - x) = 0$, and thus $\lim_n f(x_n) = f(x)$, and f is continuous.