PS/Ec 172, Homework 5
Due Friday, May 15th

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

(1) *The envelope paradox.* A coin is tossed until it comes out heads. Let k denotes the (random) number of tosses. There are two pieces of paper; on one is written the number 10^k, and on the other 10^{k+1}. One of the two notes is chosen at random and given to Rachel. The other is given to Diego. They each look at their own note. If both want to trade then they are allowed to. After trading (or not) each is given an amount of money equal to the number written on his or her paper.

Formally, the states of the world are $\Omega = \{1, 2, 3, \ldots\} \times \{0, 1\}$, where the first coordinate is the number of tosses and the second corresponds to the random allocation of notes. There is a common prior, which, for $(k, b) \in \Omega$ is

$$\mu(k, b) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = 2^{-(k+1)}.$$

Rachel's signal is $t_R(k, b) = 10^{k+b}$ and Diego's signal is $t_P(k, b) = 10^{k+1-b}$. Rachel's utility for not trading is t_R. Her utility for trading is t_P. Diego's utility for not trading is t_P, and his utility for trading is t_R.

(a) *20 points.* For each possible value of Rachel's signal, calculate her conditional expected utility for trading and for not trading. Do the same for Diego.

(b) *20 points.* Let A be the event that both want to trade. Is it common knowledge at any ω?

(c) *20 points.* Explain the apparent conceptual conflict with the no trade theorem. Which assumption in the theorem is not satisfied?

(2) *Reserve prices.* Ilya and Andrew would both like to buy an item owned by Amrita. Ilya and Andrew's valuations are chosen independently from the uniform distribution on $[0, 1]$, and each is known only to himself.

(a) *10 points.* What is Amrita's expected revenue from a second price auction?

(b) *20 points.* Amrita now introduces a *reserve price* $b_r \in [0, 1]$: if the maximum bid is under b_r then the auction is canceled, no one gets the item and no one pays. Otherwise, the winner pays the maximum of b_r and the loser's bid. What is her expected revenue, as a function of b_r?

(c) *10 points.* What is the maximal expected revenue she can get by choosing b_r optimally?

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(3) *Bonus: a riddle with both prisoners and hats (Gabay-O'Connor game).* There are n prisoners standing in a line. The first can observe all the rest. The second can observe all except the first, etc. Each is given either a red or a blue hat which he cannot see. Now, starting with the first prisoner, each in turn has to guess the color of his hat, a guess which the rest can hear.

(a) *1 point.* The prisoners are allowed to decide on a strategy ahead of time. Find one in which they all guess the color correctly, except maybe the first prisoner.

(b) *1 point.* Do the same, but for an infinite line of prisoners.

(c) *1 point.* For an infinite line of deaf prisoners, find a strategy in which at most finitely many of them guess incorrectly.