1. In the parliment of the tiny State of Consensustan there are only two senators, and a bill is passed only if there is a consensus vote for it. Before voting, each senator observes a private signal s_i, chosen independently from the uniform distribution on $[-1, 1]$. If the bill passes, both senators incur a utility of $s = s_1 + s_2$.

   Hence the game is formally defined as follows:

   (a) There are 2 players.

   (b) Each player has a type s_i. Types are independent and chosen uniformly from $[-1, 1]$. That is, the probability that s_i is below x is $(x + 1)/2$.

   (c) The strategy space is $\{Y, N\}$.

   (d) The utility when both choose Y is $s = s_1 + s_2$ for both. If at least one senator votes N, then both get utility 0.

   A threshold equilibrium is one where each senator i votes Y iff her private signal is above some threshold π_i. Symmetric threshold equilibria are those in which both the thresholds π are equal.

   Hint: the probability that $s_1 + s_2$ is below x is $(x+2)^2/8$ for $-2 \leq x \leq 0$ and is $1 - (x - 2)^2/8$ for $0 \leq x \leq 2$.

   (a) Show that in a symmetric threshold equilibrium with threshold π, the expected value of s conditioned on player 1's private signal s_1 and conditioned on player 2 voting Y is $s_1 + (1 + \pi)/2$.

   Conditioning on the second player voting yes is the same as conditioning on her having signal above π. Hence her conditional signal will be uniformly distributed between π and 1, with expectation $(1+\pi)/2$. Thus the expectation of $s = s_1+s_2$ will be $s_1+(1+\pi)/2$.

   (b) Find two symmetric threshold equilibria.

   Assume that both players have some threshold π. Then the probability that the other senator votes Y is $(1-\pi)/2$. Conditioned on the other senator voting Y, the expectation of her private signal is $(1 + \pi)/2$.

   Player i's expected utility for voting Y will thus be $(s_i+(1+\pi)/2)\cdot (1 - \pi)/2$. This is increasing in s_i, and so her best response is to vote Y above some threshold, and N below it. It is continuous, and so in that threshold she will be indifferent. For this to be an equilibrium this threshold must be π. Hence it must be that

$$(\pi + (1+\pi)/2) \cdot (1-\pi)/2 = 0.$$

The two solutions are $\pi = -1/3$ and $\pi = 1$.

(c) A consulting firm has offered the senators to reliably communicate their signals to each other for a fee of c. Assuming that the other option is to play the best symmetric threshold equilibrium, what is the maximum c that each would be willing to pay?

Assuming a symmetric threshold with $\pi = -1/3$, the probability that both vote Y is $(2/3)^2$. Conditioned on voting Y, each private signal has expectation $1/3$, and so s has expectation $2/3$. Thus each senator's expected utility is $(2/3)^3$.

With revealed signals, the probability that s is positive is $1/2$. That expectation of s conditioned on being positive is $2/3$, and so their expected utility is $1/3$. Thus they would be willing to pay $1/3 - (2/3)^3 \approx 0.04$ for this information.

2. Consider the following game played by two players. There are infinitely many time periods $\{0, 1, 2, \ldots\}$. In each time period each player has to choose whether to stay or quit.

As long as both players stay, each gets \$1 every period. If one of the players decides to quit on some period, the game ends, and each player who quit receives \$10 on that period. Thus if both quit both receive \$10. If one quits and the other stays, then the one who quit gets \$10 and the one who stayed gets nothing. Either way, neither get any more money in subsequent periods.

The players both discount utilities with factor λ, so that the utility of a player when receiving (u_t) in period t is

$$\sum_{t=0}^{\infty} \lambda^t u_t.$$

(a) Assume $\lambda < 0.9$. Find all equilibria.

When $\lambda < 0.9$, the utility for quitting immediately is 10, while the utility for staying is strictly less, regardless of the other player's actions. Thus the only equilibrium is that in which both players quit at $t = 0$.

(b) Assume $\lambda > 0.9$. Find all equilibria in which both players stay in each period with probability $p > 0$.

When $\lambda > 0.9$, assume the player is always indifferent between quitting and staying, and that the other player always stays with probability p. Then the utility for quitting is 10, while the utility for staying is $p \cdot (1 + \lambda \cdot 10)$. Since the player is indifferent then

$$p = \frac{10}{1 + 10\lambda}$$

is an equilibrium. Since $\lambda > 0.9$, $1 + 10\lambda > 10$, and so this probability is indeed between 0 and 1.

3. Alice and Bob are walking to lunch when they spot a \$7 note in a tree. They both quickly realize that the only way they can reach it is by having one of them climb on the shoulders of the other. It thus remains for them to agree on how they will divide the money between them once they retrieve it.

Alice first makes an offer to Bob. Her offer has to be one of

$$\{\$0, \$1, \$2, \$3, \$4, \$5, \$6, \$7\},$$

corresponding to the size of Bob's share.

If Bob accepts they fetch the money and split it accordingly. If Bob rejects then he makes an offer to Alice. If she accepts they fetch the money and split it accordingly. Otherwise she makes an offer again, etc. At most $T > 0$ offers can be made before they have to go to class and the game must end. If T offers are rejected then the money is left in the tree.

(a) Construct a Nash equilibrium in which they both miss lunch and receive no money.

Consider the strategy profile in which all offers are always for \$0, and no offer is ever accepted. This is a Nash equilibrium, since accepting would not increase the utility of the accepter, resulting again in utility of \$0. Likewise, changing the offer would not change the fact that it is rejected, again resulting in utility of \$0.

(b) Consider the case that $T = 17$. What are their possible utilities in subgame perfect equilibria?

Since T is odd, Alice makes the last offer. In every subgame perfect equilibrium, any last offer of \$1 or more must be accepted. When her offer is \$0, both accept (A) and reject (R) are optimal for Bob. We divide into these two cases.

In case A (i.e., when Bob accepts a \$0 offer in the last period) her unique optimal strategy in the last period is to offer \$0. She knows that she can guarantee herself \$7 by never offering more than \$0, and never accepting any offer lower than \$7, since that would bring her to the last period. Thus in this case her utility would be \$7, and Bob's would be \$0.

In case R her unique optimal strategy is to offer \$1. Thus she can, by the same argument, guarantee herself \$6, and so in this case her utility would be \$6 and Bob's would be \$1.

4. Consider a the following game with three players. Each player i has to choose an amount $a_i \in \{0, 1\}$ to invest in a public goods project. The total amount that is invested is multiplied by 2 and returned to the players, so that

$$u_i(a_1, a_2, a_3) = \frac{2}{3}(a_1 + a_2 + a_3) - a_i.$$

For each of the variants below calculate the maximum and minimum sum of utilities achievable in equilibrium. Consider subgame perfect equilibria in the sequential move setting, mixed Nash equilibria in the simultaneous move setting, and Bayes-Nash equilibria in the random order moves setting.

Note that

$$u_i(a_1, a_2, a_3) = -\frac{1}{3}a_i + \frac{2}{3}\sum_{j \neq i} a_j.$$

(a) Sequential moves with perfect information: player 1 moves first, then 2 and then 3. Each player observes the actions of her predecessors.

Sequential moves with perfect information: By the above form of the utility, given a_1 and a_2 we have that u_3 is strictly decreasing in a_3. Thus in equilibrium $a_3 = 0$. By backward induction $a_2 = 0$ and then also $a_1 = 0$.

(b) Simultaneous moves: the players choose simulaneously, without any information on the others' choices.

Simultaneous moves: again, by the above form, the only equilibrium is all zeros.

(c) Random order moves: A social planner chooses uniformly at random the order at which the players move, so that each possible order is chosen with probability 1/6. Denote the randomly chosen order by (i, j, k), so that player i moves first, then j and then k. Player i is told that she is first, and chooses her action a_i. Then player j is told that she is not first, and is also told a_i. She then chooses a_j. Finally, player k is told that she is not first, and is told a_j. She then chooses a_k. Note that players j and k both know that they are not first, but do not know if they are second or third.

Random order moves: clearly, every player choosing 0 under all circumstances is an equilibrium. There is also an equilibrium in which, on path, all players choose 1: in this equilibrium the strategies of a player are to choose 1 if either (1) this player learns that she is first, or (2) she learns that she is not first and that her predecessor chose 1. In all other cases the players choose 0. To see that this is an equilibrium, a simple calculation shows that on path the utility is 1. Off path, if the first player invests a then the rest will not invest, and she her utility will be negative. If a player who is not first does not invest then with probability 1/2 she is second, and her utility will be 2/3. With probability 1/2 she is third, and her utility will be 4/3. Thus her expected utility is 1, and so she is indifferent between the two actions.

5. Two cars arrive simultaneously at an intersection. In each time period $t \in \{0, 1, 2, \ldots\}$ each of the cars that have not yet moved choose simultaneously one of the following actions: Y (yield) or D (drive). Once a car has chosen D the game is over for that car and the driver collects her utility.

A pure strategy in this game is a complete contigency list of what to choose in $\{Y, D\}$ after each history. A behavioral stragegy consists of a distribution over $\{Y, D\}$ for each history.

The utility of a driver is minus infinity if she never drives. Otherwise, if she drives in period t, her utility is $-t$, with an additional penalty of $c > 1$ if the other driver also chose to drive in period t.

(a) Suppose both drivers choose the same strategy: as long as the other driver has not driven, drive with probability p and yield with probability $1 - p$. If the other driver has already driven, drive. Find p such that this is an equilibrium and calculate the expected utility of the players.

$p = 1/\sqrt{c}$. The utility is $-\sqrt{c}$.

(b) What is the expected utility in the best symmetric correlated equilibrium?

$-1/2$.

6. Let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a finite normal form game, so that N is a finite set of players, A_i is a finite set of actions for player i, $A = \prod_i A_i$ and $u_i \colon A \to \mathbb{R}$ is player i's utility function.

Let $\sigma \in \prod_i \Delta(A_i)$ be a mixed strategy profile. Denote by $u_i(a_i, \sigma_{-i})$ the expected utility of playe i for taking the action a_i:

$$u_i(a_i, \sigma_{-i}) = \sum_{b_{-i}\in A_{-i}} u(a_i, b_{-i}) \prod_{j\neq i} \sigma_j(b_j).$$

Fix $\beta > 0$. Let the *quantal best response* of player i to σ_{-i} be the distribution $\mathrm{BR}_i^\beta(\sigma_{-i})$ over A_i given by

$$\mathrm{BR}_i^\beta(\sigma_{-i})(a_i) = \frac{\mathrm{e}^{\beta u_i(a_i,\sigma_{-i})}}{\sum_{b_i\in A_i} \mathrm{e}^{\beta u_i(b_i,\sigma_{-i})}}$$

We say that σ is a *quantal response equilibrium* if each σ_i is a quantal best response to σ_{-i}, i.e., $\mathrm{BR}_i^\beta(\sigma_{-i}) = \sigma_i$ for all i.

(a) Fix σ_{-i} and a_i such that a_i is not a best response to σ_{-i}: there is a b_i such that $u_i(b_i, \sigma_{-i}) > u_i(a_i, \sigma_{-i})$. Show that, as β tends to infinity, $\mathrm{BR}_i^\beta(\sigma_{-i})(a_i)$ tends to zero.

By the definition of $\mathrm{BR}_i^\beta(\sigma_{-i})$,

$$\frac{\mathrm{BR}_i^\beta(\sigma_{-i})(a_i)}{\mathrm{BR}_i^\beta(\sigma_{-i})(b_i)} = \frac{\exp(\beta u_i(a_i,\sigma_{-i}))}{\exp(\beta u_i(b_i,\sigma_{-i}))}.$$

This ratio goes to zero, since the utility of a_i is smaller than that of b_i. Since the probability of b_i is at most 1, it follows that the probability of a_i tends to zero.

(b) Prove that G has a quantal response equilibrium. You can use Brouwer's Fixed Point Theorem.

Let $T_i \colon \prod_j \Delta(A_j) \to \Delta(A_i)$ be the map that assigns to σ the quantal best response $\mathrm{BR}_i^\beta(\sigma_{-i})$. Let $T\colon \prod_i \Delta(A_i) \to \prod_i \Delta(A_i)$ be the map given by $T = \prod_i T_i$. Then T is continuous. Since $\prod_i \Delta(A_i)$ is convex, T has a fixed point by Brouwer, which is a quantal response equilibrium.

7. The Chandler Cafeteria has started offering *escargot*. Lewis and Clark are eager to try it, but both are afraid that it is awful. A-priori, there is a 10% chance that it is awful (A) and a 90% chance that it is good (G).

There are time periods $t \in \{0,1,2,\ldots\}$, and in each time period they each have to simultaneously decide whether to eat (E) it or not (N). Once one of them has decided to eat it, the quality of the escargot is revealed to both and never changes.

The stage utility (at any period t) for taking action a with escargot of quality q is

$$u_t(a,q) = \begin{cases} 0 & \text{if } a = N \\ 1 & \text{if } a = E \text{ and } q = G \\ -40 & \text{if } a = E \text{ and } q = A \end{cases}.$$

A player's total utility in the game is

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t u_t$$

for $\delta = 9/10$.

(a) Explain why there are no equilibria in which neither of the players ever eat.

(b) Find a pure Nash equilibrium in which Lewis's expected utility is higher than Clark's.

(c) Find a symmetric mixed Nash equilibrium.

(d) Find a symmetric correlated equilibrium in which the players' expected utilities are higher than in the symmetric mixed Nash equilibrium.

8. Consider the following extensive form game with incomplete information, played by a student, a teacher and an employer.

   (a) The student decides whether or not to study.

   (b) Studying costs 1/3, while not studying costs nothing.

   (c) If she studies then she will be able to solve the exam.

   (d) If she does not study then she will be able to solve the exam with probability $0 < p < 1$.

   (e) She takes the exam, and solves it if she can.

   (f) The teacher decides whether to read the exam.

   (g) Reading the exam costs 1/4.

   (h) If the teacher reads the exam he knows whether it was solved.

   (i) The teacher has to give a grade: either pass or fail.

   (j) The teacher gets utility 1 from passing a solved exam or failing an exam that was not solved. Otherwise he gets utility 0.

   (k) The employer observes the grade, and gets utility 1 from hiring a student who can solve the exam, utility -1 from hiring a student who cannot, and utility of 0 from not hiring.

   (l) The student gets utility 1 from getting hired, and utility 0 from not getting hired.

   The following questions are for 20 points each.

   (a) **Harvard.** For which values of p does there exist an equilibrium in which the student **does not** study, the teacher **does not** read the exam, the teacher **passes** the student and the employer **hires** the student?

(b) **Berkeley.** For which values of p does there exist an equilibrium in which the student **does not** study, the teacher **does** read the exam, the teacher **either passes or fails** the student (depending on how she did on the exam), and the employer **hires** the student only if she passed?

(c) **Caltech.** For which values of p does there exist an equilibrium in which the student **does** study, the teacher **does not** read the exam, the teacher **passes** the student and the employer **hires** the student?