SS 205B, SET 1
DUE FRIDAY, JANUARY 14TH

Collaboration on homework is encouraged, but individually written solutions are required. Also, please name all collaborators and sources of information on each assignment; any such named source may be used.

Note: some of the claims that you are asked to prove might be false. For these claims please provide a counterexample.

(1) Let X be a subset of $\mathbb{R}^n$. Recall that $D \subseteq X$ is dense in X if for every $x \in X$ and every $\varepsilon > 0$ there is a $d \in D$ such that $\|x - d\| < \varepsilon$. Prove that every subset of $\mathbb{R}^n$ has a countable dense subset.

(2) Find a closed preference relation on $X = \mathbb{R}^2_+$ that is convex, not convex*, and for every $x \in X$ there is an $x' \in X$ such that $x' \succ x$. Or prove that no such relation exists.

(3) Consider a consumer with a closed, convex consumption set $X \subseteq \mathbb{R}^L$ and a closed preference $\preceq$ on X. Recall that given $p \in \mathbb{R}^L$ and $w \in \mathbb{R}$, we denote

$$X^*(p,w) = \{x^* \in X : p \cdot x^* \leq w \text{ and } p \cdot x \leq w \text{ implies } x^* \succeq x\}.$$

Recall also that $\preceq$ is said to be locally non-satiated (LNS) if for every $\varepsilon >$ and $x \in X$ there is a y such that $\|x - y\| \leq \varepsilon$ and $y \succ x$.

Finally, recall that $\succ$ is said to be convex if $x' \succeq x$ and $x'' \succeq x$ implies $z \succeq x$ for all $\alpha \in [0,1]$ where $z = \alpha x' + (1-\alpha)x'' \in X$.

(a) Show that if $\preceq$ is LNS, and if X is connected, then X cannot be compact. Hint: use the theorem stated in class which guarantees that $\preceq$, as a closed preference on the closed connected set X, is represented by a continuous utility function $u : X \to \mathbb{R}$.

(b) Show that if $\preceq$ is convex then $X^*(p,w)$ is convex (whenever it is non-empty).

(c) Show that if $\preceq$ is strictly convex then $X^*(p,w)$ is either empty or a singleton.

Omer Tamuz. Email: tamuz@caltech.edu.